Introduction to Combinatorics

Notes from MATC44 Lecture

Jan Miguel Marchan

Contents

Part One

1. Combinatorics

1.1 Introduction to Combinatorics

1.1.1 What is Combinatorics?

Combinatorics is a field of mathematics related to existence, proving the existence or non-existence of combinatorial objects; construction, describing how to create such objects; enumeration, computing the number of such objects; and optimization, determining which objects satisfy some extremal property.

1.1.2 Combinatorial Game Theory

Definition 1.1.1 — Combinatorial game. A *combinatorial game* is a game that has the following properties:

- Two players take turns
- There is no luck involved
- Both games have perfect information

Definition 1.1.2 — Winning strategy. *Winning strategy* is a strategy in a combinatorial game that guarantees that one player will always win regardless of what the other player does.

For example, tic-tac-toe has no winning strategy because either side can force a draw. Checkers is a mathematically solved game, that is, if each side plays perfectly, the game will end in a draw. Connect Four is also mathematically solved, and the first player has a winning strategy.

1.1.3 The Game of Nim

Nim is a two player game. There are n piles of candies. On each turn, a player chooses a pile and removes at least one candy from it. (They may remove any number of candies as long as they're in the same pile.) The player who takes the last candy wins (in normal Nim; there is a variant where the last candy taker loses).

Theorem 1.1.1 For a game of Nim with two piles of sizes m and n:
- If $n = m$, then player two can force a win.
- If $n \neq m$, then player one can force a win.

Proof. The winning strategy is to balance the two piles so they are the same size. The player who unbalances the piles can always be followed by a balancing move. In particular, taking away the last candy is a balancing move. □

Theorem 1.1.2 For a game of Nim with three piles of sizes m, n and n, then player 1 can force a win.

Proof. The player 1 can remove the pile of size m, then they become player 2 in two pile Nim where the two piles are balanaced. Then by Theorem 1.1.1, that player wins □

We can convert the size of each pile into binary, then consider each digit place.

Definition 1.1.3 — Nim-sum. The *Nim-sum* is the sum of all piles in binary without carry.

You can also think of it as an exclusive or in each digit place.

■ **Example 1.1** Consider the Nim-sum of $7 \oplus 9 \oplus 12 \oplus 15$

7	0111
9	1001
12	1100
15	1111
	1101

We want to change the Nim-sum to 0, so we could remove 13. ■

Theorem 1.1.3 — Nim Theorem. The winning strategy in (normal) Nim is to finish every move with a Nim-sum of zero

Proof. The winning strategy is to have a total even number of 1s in each binary digit place.

If the table is balanced, then the next move will unbalance it. You will always take away at least one candy from one pile, so there will always be a 1 changing to a 0. So a balanced table will always be unbalanced by the next move.

If the table is unbalanced, then you can always balance it with one move. Take the most significant bit in the "Nim-sum" and pick a pile with a 1, then remove candies to balance. □

1.1.4 Graph Theory

The Seven Bridges of Königsberg is a problem that asks if it is possible to cross all seven bridges exactly once in a singular path.

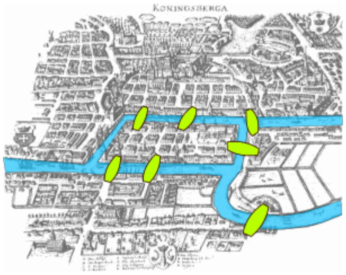


Figure 1.1

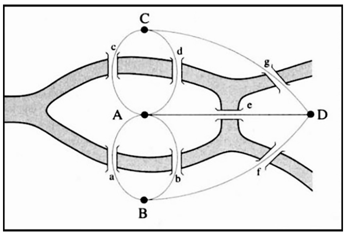


Figure 1.2

It is impossible since every vertex has an odd number of edges coming out of it.

In graph theory, we only care about the structure of the graph, not the geometry, i.e., the connections between nodes.

1.2 Proof Techniques and Problem Solving

1.2.1 Pigeonhole principle

Theorem 1.2.1 — Pigeonhole principle. If n pigeons are placed into m pigeonholes and $n > m$, then there exists a pigeonhole with at least two pigeons.

Proof. Suppose n pigeons are placed into m pigeonholes such that $n > m$.
For contradiction, assume that every pigeonhole contains at most one pigeon.
Then there are at most m pigeons since there are m pigeonholes.
But we assumed that there are n pigeons and that $n > m$, so this is impossible.
Therefore, there must be a pigeonhole with more than one pigeon. □

The pigeonhole principle is not constructive (i.e., does not specify a pigeonhole). It allows us to count objects with a common property.

An alternative statement of the pigeonhole principle is the following:

If $f : X \to Y$ is a function and $|X| > |Y|$, then there exists $y \in Y$ and distinct $x, x' \in X$ such that $f(x) = f(x') = y$ (i.e., f is not injective).

Theorem 1.2.2 — Generalized pigeonhole principle. Let m and k be positive integers. If $mk + 1$ pigeons are placed into m pigeonholes, then there exists a pigeonhole with at least $k + 1$ pigeons.

R $mk+1$ is a "sharp" (i.e., "tight") lower bound.

Definition 1.2.1 — Ceiling function. The *ceiling function*, denoted by $\lceil x \rceil$ maps the x to the least integer greater than or equal to x, $\lceil x \rceil = min\{n \in \mathbb{Z} : n \geq x\}$.

It is straightforward to verify that $x \leq \lceil x \rceil < x+1$ holds for any $x \in \mathbb{R}$.
The following is another version of the generalized pigeonhole principle:

Theorem 1.2.3 Let N and k be positive integers. If N objects are distributed to k boxes, then at least one of the boxes must hold $\lceil N/k \rceil$ objects.

R Make sure you know how to *distribute* the objects into the boxes, and not the boxes to the objects.

Proof. Suppose N objects are distributed to k boxes.
For contradiction, assume that every box contains at most $\lceil N/k \rceil - 1$ objects.
Then there are at most $k(\lceil N/k \rceil - 1)$ objects since there are k boxes.
By using $\lceil x \rceil < x+1$ with $x = N/k$, we have

$$\text{\# of objects} \leq k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\frac{N}{k} + 1 - 1\right) = N$$

But we assumed that there are N objects, so this is impossible.
Therefore, there must be a box with $\lceil N/k \rceil$ objects. □

We start with an introductory problem.

■ **Example 1.2** At least 40 students at UTSC share the same birthday. ■

Proof. There are 14,547 students at UTSC. There are 366 possible birthdays (when including the leap day).
By the pigeonhole principle, there is a birthday which is shared by $\lceil 14,547/366 \rceil = 40$ students. □

■ **Example 1.3** Every subset of size 6 of the set $\{1,2,3,4,5,6,7,8,9\}$ contains two elements whose sum is 10. ■

The problem is finding the pigeons and pigeonholes.

Proof. Let $A \subseteq \{1,2,\ldots 9\}$ with $|A| = 6$.
Consider the subsets S_i of $\{1,9\},\{2,8\},\{3,7\},\{4,6\},\{5\}$ as the pigeonholes. Then the elements of A are pigeons.
Distribute the elements of A to S_i according to their label. There are 6 pigeons and 5 pigeonholes.
By pigeonhole principle, there is a 2-element subset with 2 elements from A, hence, we have sum 10 since all the 2-element subsets sum to 10. □

■ **Example 1.4** Every subset of size $n+1$ of $\{1,2,\ldots,2n\}$ has two elements that are consecutive. ■

Proof. Consider the subsets S_i of $\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$ as the pigeonholes. Let $A \subseteq \{1,2,\ldots n\}$ with $|A| = n+1$.
Distribute the elements of A to S_i according to their label. There are $n+1$ pigeons and n

pigeonholes.

By pigeonhole principle, there is a 2-element subset with 2 elements from A, hence, we have two consecutive numbers in the subset. □

Exercise 1.1 What is the smallest N such that given any N positive integers, there are two whose sum or difference is divisible by 100?

The answer is $N = 52$.

If $N = 51$, then we have $\{1,2,\dots 50,100\}$ which has no sum or difference divisible by 100.

Proof. Create 51 pigeonholes $\{00\},\{01,99\},\{02,98\},\dots,\{49,51\},\{50\}$.

Then consider the N positive integers as the pigeons. We distribute them into the pigeonholes by considering them modulo 100.

By the pigeonhole principle, two of the 52 integers, say x and y, must belong to the same set. Either x and y have the same last two digits and their difference ends in 00, or they have different last two digits and their sum ends in 00. □

1.2.2 Problem Solving Tips

To solve a problem, some things we can do are:

- plug in numbers
- look for patterns
- draw pictures
- introduce notation
- think about "parity"
- look for symmetry
- divide into cases
- modify the problem (reduce or generalize)

1.2.3 Induction

To prove a family of statements $P(k)$, we prove some base cases $P(1),\dots$. Then, we show that $P(1),\dots,P(n) \Rightarrow P(n+1)$.

Exercise 1.2 Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof. We prove this by induction.

Base Case: Let $n = 1$. Then $\sum_{i=1}^{n} i = 1 = \frac{1(2)}{2} = \frac{n(n+1)}{2}$.

Inductive Hypothesis: Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$.

Inductive Step: We want to show that $\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$.

$$
\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) && \text{(By IH)} \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}
\end{aligned}
$$

$\therefore \sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$.
$\therefore$ By induction, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$ □

Exercise 1.3 Prove chicken nuggets are sold in packages of sizes 4, 6, 9, and 20. Prove that for $n \geq 24$, we can buy exactly n nuggets by buying some combination of packages.

Proof. We prove this by induction.
BASE CASE: Let $n = 24$. Then $n = 4+4+4+4+4+4$.
Let $n = 25$. Then $n = 4+4+4+4+9$.
Let $n = 26$. Then $n = 4+4+9+9$.
Let $n = 27$. Then $n = 9+9+9$.
INDUCTIVE HYPOTHESIS: Assume we can buy exactly n nuggets for any $24 \leq k \leq n$ where $n \geq 28$.
INDUCTIVE STEP: We want to show we can buy exactly $n+1$ nuggets.
By IH, we can buy exactly $k = (n+1)-4$ nuggets since $24 \leq (n+1)-4 \leq n$.
$\therefore$ We can buy $n+1$ nuggets because we can buy $k = (n+1)-4$ nuggets and a 4 pack. □

The largest number which cannot be expressed as a (non-negative) linear combination of a set of given numbers is the *Frobenius number*.

1.2.4 Coloring/parity proof

The idea of coloring/parity proofs is to partition a set into a finite number of subsets by colouring each element of the subset by the same colour.

In 1961, M.E. Fisher showed that an 8×8 chessboard can be covered by a 2×1 dominoes in $2^4 \times 901^2 = 12,988,816$ ways.

Exercise 1.4 Cut out two opposite corners of a chessboard. How many ways can you cover the 62 squares using 31 dominoes?

There are 0 ways to cover the 62 squares using 31 dominoes.

Proof. Fact: Every domino must cover one black and one white tile.
Consider the "mutilated" chessboard, which is a chessboard with two opposite corners removed.
These opposite corners have the same color.
WLOG assume the removed opposite squares are black. Then the mutilated chessboard has 30 black and 32 white squares.
To derive a contradiction, assume there exists a tiling.
By Fact, the dominoes must cover exactly 31 black and 31 white squares. Contradiction.
$\therefore$ There is no tiling of the 62 square mutilated chessboard by 2×1 dominoes. □

It doesn't matter that we removed opposite corners.

Theorem 1.2.4 There is no tiling using dominoes of a chessboard where two squares of the same color are removed.

The proof of this is the same as the last proof.
What if we remove two squares of different colors instead?

Theorem 1.2.5 [Gomory, 1973] An 8×8 chessboard with one black and one white square removed can always be covered by exactly 31 2×1 dominoes.

We create a cycle around the chessboard.

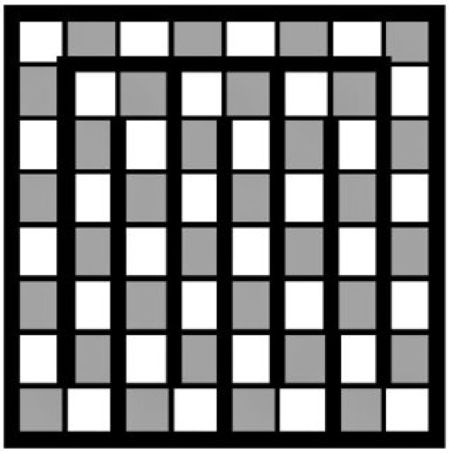


Figure 1.3

Proof. By removing oppositely colored squares, the cycle in Figure 1.3 splits into two paths of even length, i.e., they have an equal number of black and white squares. Then both paths can be covered. □

1.2.5 Invariance principle

One of the proof techniques that we have is to look for things that don't change.

Exercise 1.5 Divide a circle into six sectors and label the sectors by 1,0,1,0,0,0. Every minute, you may increment two neighbors (sectors which share an edge) by 1. Is it possible to make all numbers equal in finite time?

We divide the circle as in 1.4.

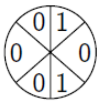


Figure 1.4

It is not possible to make all numbers equal.

Proof. Let the sectors of the circle be $a_1, a_2, a_3, a_4, a_5, a_6$

Let $\mathcal{I} = a_1 + a_3 + a_5 - a_2 - a_4 - a_6$.

Claim: $\mathcal{I}$ is invariant (i.e., doesn't change).

Proof. Each minute (or step) adds to $\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_5\}, \{a_5, a_6\}, \{a_6, a_1\}$. So the value of $\mathcal{I}$ does not change. □

We want for $i = 1, \ldots, 6$ that $a_i = m$ for some $m \geq 0$. So $\mathcal{I} = m + m + m - m - m - m = 0$.

Since we start with the state such that $\mathcal{I} = 2$ and $\mathcal{I}$ is invariant (by Claim), then it is impossible to reach the state with all numbers equal. □

2. Counting

2.1 Counting Principles

2.1.1 Addition and Multiplication Principle

Theorem 2.1.1 — Addition Principle. Suppose that for $i = 1, \dots, k$, there are n_i ways for event E_i to occur.

If the ways the different events can occur are pairwise disjoint, the number of ways for at least one of the events $E_1, E_2, \dots, E_k$ to occur is

$$\sum_{i=1}^{k} n_i$$

This can be alternatively be formulated in the following way:
Let $A_1, A_2, \dots, A_k$ be any k finite sets. If $A_i \cap A_j \neq \varnothing$ for all $1 \leq i, j \leq k$ with $i \neq j$, then

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{i=1}^{k} |A_i|$$

Theorem 2.1.2 — Multiplication Principle. Suppose that event E can be decomposed into k ordered events such that for $i = 1, \dots, k$, there are n_i ways for event E_i to occur.

If the ways the different events are pairwise disjoint, the number of ways for E to occur is

$$\prod_{i=1}^{k} n_i$$

This can be alternatively be formulated in the following way:

Let $A_1, A_2, \dots, A_k$ be any k finite sets and

$$\prod_{i=1}^{k} A_i := \{ (a_1, a_2, \dots, a_k) \mid a_i \in A_1, a_2 \in A_2, \dots a_k \in A_k \}$$

be the Cartesian product of $A_1, A_2, \dots, A_k$. Then $|\prod_{i=1}^{r} A_i| = \prod_{i=1}^{r} |A_i|$.

Intuitively, if we have n_1 ways to do one thing and n_2 ways to do something else, then there are $n_1 + n_2$ to choose exactly one of the actions and $n_1 \cdot n_2$ ways to do both.

■ **Example 2.1** To reach city D from A, we must pass through cities B and C, such that there are 2 ways from A to B, 4 ways from B to C, 2 ways from C to D.

Then by multiplication principle, there are $2 \times 4 \times 2 = 16$ ways to go from A to D. ■

Exercise 2.1 Find the number of positive divisors of the number 600 (including 1 and 600).

We use the fact that any positive divisor of $n \in \mathbb{N}$ is a product of prime divisors of n raised to some power.

Solution. First, we get the prime factorization of $600 = 2^3 \cdot 3^1 \cdot 5^2$.

Then any positive integer m is a divisor of 600 iff m is of the form $2^a \cdot 3^b \cdot 5^c$ for $0 \leq a \leq 3, 0 \leq b \leq 1, 0 \leq c \leq 2$.

So the number of positive divisors of 600 is the number of triples (a,b,c) where $a \in \{0,1,2,3\}$ and $b \in \{0,1\}$ and $c \in \{0,1,2\}$, which by multiplication principle is $4 \times 2 \times 3$. □

We can generalize to a function $\sigma(n)$ to some product of the magnitude of the prime powers plus 1.

Exercise 2.2 Let $X = \{1,2,\dots,100\}$ and $S = \{ (a,b,c) \mid a,b,c \in X, a < b, a < c \}$.

Find $|S|$.

Solution. We can split up into 100 disjoint cases by considering $a = 1,2,\dots,100$.

Fix k such that $1 \leq k \leq 100$.

Then we count (a,b,c) such that $a < b, a < c$.

So the number of choices for b is $100 - k$. Also, the number of choices for c is $100 - k$.

Then the number of triples of (k,b,c) is $(100-k) \times (100-k)$ by multiplication principle.

By addition principle, there are $|S| = 99^2 + 98^2 + \cdots + 1^2 + 0^2 = \sum_{i=0}^{99} i^2 = \frac{99(99+1)(2(99)+1)}{6} = 328,350$. □

2.2 Permutations and factorial

Definition 2.2.1 — Permutation. A *permutation* is an arrangement of distinct objects.

Definition 2.2.2 — Factorial. The *factorial* of n objects is defined as

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

with the convention that $0! = 1$.

Alternatively, we can say that $n! = n(n-1)!$ with $1! = 1$ and $0! = 1$ for $n \in \mathbb{Z}_{\geq 0}$.

Theorem 2.2.1 (i) There are $n!$ permutations of a set of n objects.
(ii) Fix k so that $0 \leq k < n$. The number of permutations of k objects of a set of size n is

$$n \cdot (n-1) \cdot (n-2) \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

As a proof outline, we have n choices for the first slot, $n-1$ choices for the second slot, so on, until we have $n-k+1$ choices for the kth slot.

Notation 2.1. *$P(n,k)$ is the number of arrangements of k objects of a set of size n.*

R We say that if $k > n$, then $P(n,k) = 0$.

2.3 Combinations

Definition 2.3.1 — Binomial coefficient. Let $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$.
The *binomial coefficient* is denoted $\binom{n}{k}$.
Then $\binom{n}{k}$ is the number of ways to choose k objects from a collection of n objects.
Alternatively, $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

Proposition 2.3.1 Let $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq n$. Then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

To show this, verify $P(n,k) = \binom{n}{k}k!$ using Multiplication Principle.

2.3.1 Problem Solving

	Order significant	Order not significant
Repetitions	n^k	$\binom{n+k+1}{k}$
No repetitions	$n(n-1)\cdots(n-k+1)$	$\binom{n}{k}$

The following keywords indicate problems where order matters:
- arrangement
- row
- queue
- tuple
- list
- word
- PIN/password
- committee positions

The following keywords indicate problems where order doesn't matter:
- pile
- set
- group
- committee
- bag

People are distinct since they have names.

Exercise 2.3 There are seven professors and three students in a gathering. How many ways can they be arranged in a row so that the three students form a single block.

Solution. First note that order matters, and that the ten people are distinct.

Since three students should sit together, we can treat them as a single entity.

So there are 8! ways of arranging the seven professors and the one group of three students. Then there are 3! ways of arranging those three students. So we have $8! \cdot 3!$. possibilities. □

Exercise 2.4 There are seven professors and three students in a gathering. How many ways can they be arranged in a row so that the end positions are occupied by professors and no two students are adjacent.

Solution. First, consider the arrangements of professors.

There are 7! ways to arrange the 7 professors in a row.

Fix an arbitrary one of these arrangements.

$$P_1P_2P_3P_4P_5P_6P_7$$

Then there are 6 spaces available for the students.

$$P_1\_P_2\_P_3\_P_4\_P_5\_P_6\_P_7$$

We have $\binom{6}{3}$ places to place the students and 3! ways to arrange them. So $7! \cdot \binom{6}{3} \cdot 3!$. □

Exercise 2.5 Fix $n \geq 2$.

(a) How many ways can we choose 2 people from among n people?

(b) How many ways can we partition n people into a set of size 2 and a set of size $n-2$?

Solution. (a) The answer is $\binom{n}{2}$. The set of chosen people is a special set (the "chosen set"). That is, 2 people are chosen and $n-2$ are not chosen.

(b) The answer depends on n. For example, consider $n = 4$. Then $\{\{1,2\},\{3,4\}\} = \{\{3,4\},\{1,2\}\}$, i.e., these two partitions are the same.

Thus, when $n = 4$, there are $\frac{1}{2}\binom{4}{2} = 3$ partitions.

When $n \neq 4$, the answer is $\binom{n}{2}$ because the two partitions are different sizes.

□

The takeaway is that we have to be careful of overcounting in equal size sets in partitions.

2.4 Double Counting

There are two main methods to prove combinatorial formulas:

- Algebriac proof: Use $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- Combinatorial proof: Interpret $\binom{n}{k}$ as the number of ways of choosing a committee of k people from n people.

■ **Example 2.2** Let n be a positive integer and $0 \leq k \leq n$. Prove $\binom{n}{k} = \binom{n}{n-k}$. ■

Determine a question that the identity answers. Answer the question in two different ways ("double counting"). Since both answers count the same thing, they must be equal.

Proof. QUESTION: In how many ways can we select k toys from a box of n toys?

ANSWER 1: By definition of binomial coefficient $\binom{n}{k}$.

ANSWER 2: We can pick k toys by choosing toys we don't want. We choose $n-k$ toys to discard and keep the remaining. This can be done in $\binom{n}{n-k}$ □

■ **Example 2.3** Let $n \in \mathbb{Z}^+$. Give a combinatorial proof of $n^2 = (n-1)^2 + 2(n-1) + 1$. ■

Proof. We consider the following problem:

PROBLEM: Count the number of ordered pairs (i, j) with $1 \leq i, j \leq n$.

ANSWER 1: There are n choices for i and n choices for j. Thus, there are n^2 possible ordered pairs.

ANSWER 2: Note that $+$ usually means to split into disjoint cases.

We partition the pairs according to the number of 1's in it.

- When there are no 1s, there are $(n-1)^2$ pairs.
- When there is one 1, there are $2(n-1)$ pairs, 2 choices for 1 and $n-1$ choices for the rest.
- When there are two 1s, there is only 1 pair.

□

■ **Example 2.4** Let n be a positive integer. Give a combinatorial proof of

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

. ■

Proof. PROBLEM: A pretzel shop offers n toppings. How many pretzels can you make using any number of the n toppings where each topping is used at most once?

ANSWER 1: For each of the n toppings, you have two choices, include or omit it. This gives 2^n possible combinations.

ANSWER 2: Fix k such that $0 \leq k \leq n$ and consider a pretzel with exactly k toppings. Then there are $\binom{n}{k}$ pretzels with exactly k toppings. So there are a total of $\sum_{k=0}^{n} \binom{n}{k}$ □

■ **Example 2.5** Use combinatorial proofs to prove the following:

(a) $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$

(b) $n^2 = \sum_{i=1}^{n} 2n - 1$

(c) $\sum_{k=1}^{n} n\binom{n}{k} = n2^{n-1}$

■

Proof.

(a) PROBLEM: How many ways can you take 2 marbles out of m red marbles and n blue marbles?

ANSWER 1: We know this is $\binom{n+m}{2}$.

ANSWER 2: Choose 2 red, $\binom{m}{2}$, 2 blue $\binom{n}{2}$, or 1 red and 1 blue $\binom{m}{1}\binom{n}{1} = mn$.

(b) Consider ordered pair (i, j) where you consider $k = \max i, j$.

(c) PROBLEM: Given n people, how many ways are there to select a committee of any size from 1 to n with a president?

□

2.5 Stars and Bars

Exercise 2.6 How many strings are there using four S's and two B's?

Proof. From 6 letters, choose 2 locations for B. This gives $\binom{6}{2}$ □

Exercise 2.7 How many ways can we distribute four dimes among three people?

Proof. There is a bijection between the distributions of dimes, and arrangements of 4 stars and 2 bars. So it suffices to count the number of ways to arrange 4 stars and 2 bars. This gives $\binom{6}{2}$ possibilities. □

Theorem 2.5.1 — Stars and bars. Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to partition n identical objects into m labelled groups is $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m+1}{n}$.

Proof. Set up a bijection with arrangements of stars and bars (where the stars are the objects and the bars are the separators between the groups of objects).

Because we want m groups, we require $m-1$ bars for the partition:

group 1 | group 2 | ··· | group m

We now arrange the n stars and $m-1$ bars in a row. There are a total of $n+m-1$ symbols to arrange. To count the number of arrangements, we choose the positions of the bars (or equivalently, the stars) in the row.

$\therefore$ There are $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$ arrangements. □

Observe that if we took all the partitions from Exercise 2.7, we get the equivalent: every possible way to write the number four as an ordered sum of three non-negative integers.

That is, we're enumerating the non-negative integer solutions to $x_1 + x_2 + x_3 = 4$ where $x_1, x_2, x_3 \geq 0$.

Theorem 2.5.2 Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to write n as an ordered sum of n non-negative integers (i.e., the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_m$) is $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m+1}{n}$.

The proof is just a bijection through stars and bars.

Exercise 2.8 How many ways can we distribute four dimes among three people so that every person gets at least one dime?

Proof. Note that we can't have two adjacent bars or a bar in the end psitions.

First, we place all the bars, then allocate bars to the spaces between them.

*_*_*_*

Then observe that there are only 3 places to place the 2 bars, giving $\binom{3}{2}$ possibilities. □

Theorem 2.5.3 — Stars and bars, nonempty. Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to partition n identical objects into m labelled nonempty groups is $\binom{n-1}{m-1}$.

We can think of it as stars and bars but we give m objects first.

Proof. Set up a bijection with arrangements of stars and bars (where the stars are the objects and the bars are the separators between the groups of objects).

Because we want m groups, we require $m-1$ bars for the partition:

group 1 | group 2 | ⋯ | group m

We now arrange the n stars and $m-1$ bars in a row. To count the number of arrangements, first place stars, then choose spots for bars between them (of which there are $m-1$).

$\therefore$ There are $\binom{n-1}{m-1}$. □

We get a similar theorem to Theorem 2.5.2

Theorem 2.5.4 Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to write n as an ordered sum of n positive integers (i.e., the number of positive integer solutions to $x_1 + x_2 + \cdots + x_m$) is $\binom{n-1}{m-1}$.

Exercise 2.9
(a) $x_1 + x_2 + x_3 + x_4 = 7$ with $x_i \geq 0$.
(b) $x_1 + x_2 + x_3 + x_4 = 7$ with $x_i > 0$.
(c) $x_1 + x_2 + x_3 + x_4 = 7$ with $0 \leq x_i \leq 9$.
(d) $x_1 + x_2 + x_3 + x_4 \leq 7$ with $0 \leq x_i \leq 9$.
(e) $x_1 + x_2 + x_3 + x_4 \leq 15$ with $x_i \geq -10$.
(f) $x_1 + x_2 + x_3 + x_4 \leq 13$ with $0 \leq x_i \leq 9$.

Proof.
(a) By stars and bars, $\binom{7+4-1}{4-1} = \binom{10}{3}$.
(b) By stars and bars, $\binom{7-1}{4-1} = \binom{6}{3}$.
(c) Equivalent to (a) because x_i cannot be greater than 7.
(d) Note, this is equivalent to $x_1 + x_2 + x_3 + x_4 \leq 7$ with $x_i \geq 0$.
Then, we add a slack variable to get an equivalent equation, $x_1 + x_2 + x_3 + x_4 + x_5 = 7$ with $x_i \geq 0$.
By stars and bars, $\binom{7+5-1}{5-1} = \binom{11}{4}$.
(e) Substitute $y_i = x_i + 10$. Then we have $y_1 - 10 + y_2 - 10 + y_3 - 10 + y_4 - 10 \leq 15$ with $y_i \geq 0$.
We add a slack variable y_5 to get the equality $y_1 - 10 + y_2 - 10 + y_3 - 10 + y_4 - 10y_5 = 15$ with $y_i \geq 0$.

□

2.6 Arrangements with Repetition

Say that we want the number of arrangments for letters MIKE. We could just say that there's 4! arrangements. But we can also consider that there are $\binom{4}{1}$ positions for M, $\binom{3}{1}$ for I, $\binom{2}{1}$ for K, and $\binom{1}{1}$. This lets us generalize to repetition.

■ **Example 2.6** How many arrangements are there of the letters of TORONTO? ■

Proof. There are 7 letters. From 7 empty spots, choose 3 for O's. Then, from 4 empty spot, choose 2 for T's. From 2 empty spots, choose 1 for R. From 1 empty spot, choose 1 for N.

The answer is

$$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{7!}{3!4!}\,\frac{4!}{2!2!}\,\frac{2!}{1!1!}\,\frac{1!}{1!1!} = \frac{7!}{3!2!1!1!}$$

(Alternatively, there are 7! arrangements of letters, but we have to remove what we over-counted, $\frac{7!}{3!2!}$) □

■ **Example 2.7** How many ways to arrange RRWWGGG?

How many ways to arrange a flag with 7 vertical stripes, 2 red, 2 white, and 3 green?

We have three types of breakfast food: raisin bran, waffles and grapefruit. If there are 2 bowls of raisin bran, 2 plates of waffles and 3 bowls of grapefruits available, in how many ways can we distribute them among 7 people? ■

These questions are equivalent, giving $\frac{7!}{2!2!3!}$. For the last question, fix the people in a row, then the selection of breakfast foods is just an arrangement again.

Definition 2.6.1 — Multinomial coefficient. Let n be a positive integer and $n_1, n_2, \dots, n_k$ be non-negative integers such that

$$n_1 + n_1 + \cdots + n_k = n$$

The *multinomial coefficient* is defined as

$$\binom{n}{n_1, n_2, \dots, n_k} := \frac{n!}{n_1! n_2! \cdots n_k!}$$

Theorem 2.6.1 If there are $n_i \geq 1$ objects of type i for $1 \leq i \leq k$, and there are $n = n_1 + n_2 + \cdots n_k$ objects in total, then the number of arrangements of these n objects is $\binom{n}{n_1, n_2, \dots, n_k}$.

We could switch the theorem to be the definition and the formula to be a theorem.

Proof. Generalize the process in the previous examples to get

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

arrangements and simplify to $\frac{n!}{n_1!n_2!\cdots n_k!}$ which by definition is $\binom{n}{n_1, n_2, \dots, n_k}$. □

Note that multinomial coefficients generalize binomial coefficients: $\binom{n}{k} = \binom{n}{k, n-k}$.

Also, the multinomial coefficient is a natural number, by the previous theorem. (This is why the combinatorial definition is preferred.)

■ **Example 2.8** Let k be a positive integer. Prove that $(4k)!$ is a multiple of $2^{3k} \cdots 3^k$. ■

Proof. Count the number of arrangements of elements of the multiset

$$\{a_1, a_2, a_3, a_4,\ a_2, a_2, a_2, a_2,\ \dots,\ a_k, a_k, a_k, a_k\}$$

For $i = 1, 2, \dots, k$, each a_i appears four times. The total number of elements including repears is

$$n = n_1 + n_2 + \cdots + n_k = 4 + 4 + \cdots + 4 = 4k$$

By the theorem, the number of arrangements of elements in the multiset is

$$\binom{n}{n_1, n_2, \dots, n_k} = \binom{4k}{4, 4, \dots, 4} = \frac{(4k)!}{4! \cdot 4! \cdots 4!} = \frac{(4k)!}{(4!)^k} = \frac{(4k)!}{()2^3 \cdots 3)^k} = \frac{(4k)!}{2^{3k} \cdots 3^k}$$

This is a natural number by previous observation. □

■ **Example 2.9** How many arrangements of FLIBBERTIGIBBET have no two vowels consecutive? ■

Proof. There are 4 B's, 2 T's, 1 F, 1 G, 1 L, 1 R, and 2 E's and 3 I's.

First arrange the consonants. There are $\binom{10}{4,2,1,1,1,1} = \frac{10!}{4!2!}$ ways to do this.

Then there are 11 locations for the 5 vowels, $\binom{11}{5}$. Then you have to arrange the vowels. There are $\binom{5}{2,3}$ ways to do this.

The final answer is $\binom{10}{4,2,1,1,1,1}\binom{11}{5}\binom{5}{2,3}$. □

2.7 Binomial Theorem

Notation 2.2. *For any $k,n \in \mathbb{Z}$ with $n \geq 0$, we let $\binom{n}{k} = 0$ for $0 \leq k \leq n$.*

We call $\binom{n}{k}$ a binomial coefficent because it shows up as a coefficient in the expansion of the binomial expression $(x+y)^n$.

Theorem 2.7.1 For any integer $n \geq 0$, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

One technique is to apply induction. We will use a combinatorial proof.

Proof. We want to show the coefficient of $x^{n-k}y^k$ of $(x+y)^n$ is $\binom{n}{k}$.

To expand $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}}$, we choose either x or y from each factor of $(x+y)$ and then multiply together.

To form a term with $x^{n-k}y^k$, we first select k of the n factors $(x+y)$ and pick y from these chosen factors (followed by picking x from the remaining $n-k$ factors).

The first step can be done in $\binom{n}{k}$ ways, and the second step in $\binom{n-k}{n-k} = 1$ way.

$\therefore$ The number of ways to form an $x^{n-k}y^k$ is $\binom{n}{k}$. □

2.7.1 Multinomial Theorem

Consider a trinomial cubed, $(x_1+x_2+x_3)^3$. How many ways are there to form a $x_1^2x_2$ term? We can select x_1,x_1,x_2 from the first, second, and third factors, or x_1,x_2,x_1, or x_1,x_1,x_2, giving three ways. This is equal to the number of arrangements of x_1,x_1,x_2 where $n_1 = 2$, $n_2 = 1$ and $n = 3$, i.e., $\binom{3}{2,1}$.

Theorem 2.7.2 — Multinomial Theorem. Let n be a positive integer. For all $x_1,x_2,\ldots,x_m$, we have

$$(x_1+x_2+\cdots+x_m)^n = \sum \binom{n}{n_1,n_2,\ldots,n_m} x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$$

where the summation is over all the non-negative integer solutions $(n_1,n_2,\ldots,n_m)$ of $n_1+n_2+\cdots+n_m = n$.

We can rewrite this as $\sum\limits_{n_1+\cdots+n_m} \binom{n}{n_1,\ldots,n_m} \prod_{i=1}^{m} x_i^{n_i}$

■ **Example 2.10** Find the coefficient of $x^{99}y^{60}z^{14}$ in $(2x^3+y-z^2)^100$ ■

Proof. Let $x_1 = 2x^3, x_2 = y, x_3 = -z^2$. Then by the multinomial theorem, the terms are off the form

$$\binom{100}{n_1,n_2,n_3}(2x^3)^{n_1}y^{n_2}(-z^2)^{n_3} = \binom{100}{n_1,n_2,n_3}(2x^3)^{n_1}y^{n_2}(-1)^{n_3}(z^2)^{n_3}$$

The term $x^{99}y^{60}z^{14}$ arises when $n_1 = 33, n_2 = 6, n_3 = 7$ which has coefficient

$$\binom{100}{33,60,7}2^3 3(-1)^7 = -\binom{100}{33,60,7}2^3 3$$

□

■ **Example 2.11** Prove using the binomial theorem:

() $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}$

() $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$

() $\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$

■

Proof. (a) Consider $(1+(-1))^n$. Then $1+(-1))^n = 0$ and also

$$\begin{aligned}(1+(-1))^n &= \sum_{k=0}^{n} \genfrac{}{}{0pt}{}{n}{k} 1^{n-k}(-1)^k \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}\end{aligned}$$

(b) Consider $(1+y)^n$.

$$\begin{aligned}(1+y)^n &= \sum_{k=0}^{n} \frac{n}{k} 1^{n-k} y^k \\ &= \sum_{k=0}^{n} \frac{n}{k} y^k \\ \frac{d}{dy}(1+y)^n &= \frac{d}{dy}\sum_{k=0}^{n} \frac{n}{k} y^k \\ n(1+y)^{n-1} &= \frac{d}{dy}\sum_{k=0}^{n} \frac{n}{k} k y^{k-1}\end{aligned}$$

When we set $y = 1$, we get the result $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

(c) Consider $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. We can take the definite integral from $[0,1]$.

□

2.7.2 Pascal's Triangle

Definition 2.7.1 — Pascal's Triangle. *Pascal's Triangle* is the triangular array so that the entry in the nth row and kth column is $\binom{n}{k}$, with top row as the 0th row.

Proposition 2.7.3 The sides of Pascal's triangle are equal to 1, and all other entries are the sum of the two entries above it.

We can write this mathematically as follows.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Figure 2.1: Pascal's Triangle with 7 rows

Theorem 2.7.4 For any $n \geq 2$ and $1 \leq k \leq n-1$, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. We consider the following problem for a combinatorial proof.

QUESTION: How many k element subsets S are there of $\{1,2,\dots,n\}$?

ANSWER 1: There are $\binom{n}{k}$ subsets of $\{1,\dots,n\}$ of size k by definition of binomial coefficients.

ANSWER 2: We partition according whether 1 is in S.

The number of subsets of size k with $1 \in S$ is $\binom{n-1}{k-1}$.

The number of subsets of size k with $1 \notin S$ is $\binom{n-1}{k}$. ☐

Proposition 2.7.5 (a) The entries are symmetric with respect to the middle line, i.e., $\binom{n}{k} = \binom{n}{n-k}$.

(b) The sum of each row is 2^n, where n is the level number, i.e., $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & \underline{1} & & & & \\
 & & & 1 & & 3 & & \underline{3} & & 1 & & & \\
 & & 1 & & 4 & & \underline{6} & & 4 & & 1 & & \\
 & 1 & & 5 & & \underline{10} & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & \underline{20} & & 15 & & 6 & & 1
\end{array}$$

Figure 2.2: Hockey Stick Identity, $1+3+6+10=20$

Theorem 2.7.6 — Hockey Stick Identity. For $n,k \in \mathbb{N}$ and $n \geq k$, we have $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$

That is,

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

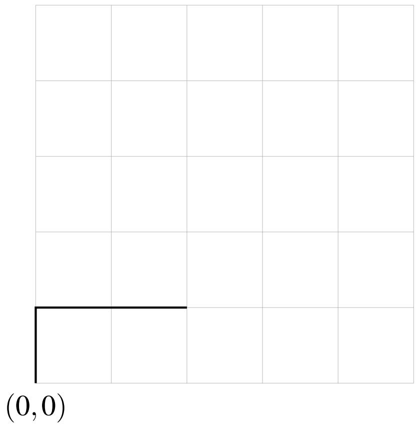
The following is a corollary, hockey stick identity with $k = 1$.

Proposition 2.7.7 The sum of the first n positive integers is

$$\sum_{i=1}^{n} i = \binom{n+1}{2}$$

These are called the *triangular numbers*.

■ **Example 2.12 — Shortest routes.** How many shortest paths (along the integer grid) from $(0,0)$ to (m,n)? ■



Proof. You need to move right m times and up n times. This is equivalent to $\binom{m+n}{m}$. □

3. Graph Theory

3.1 Graphs

Informally, a graph G is an object consisting of a collection V (or $V(G)$) of dots, called vertices, and a collection E (or $V(E)$) of lines, called edges, where every edge connects two dots.

Vertices are often denoted by lowercase letters $\{a,b,c,\ldots,v_1,v_2,v_3,\ldots\}$ and sometimes subscripts, and edges are often denoted as $\{a,b\}$ (or ab).

Definition 3.1.1 — Graph. A *graph* is a pair $G=(V,E)$, where V is a non-empty set and E is a set of two-element subsets of V.

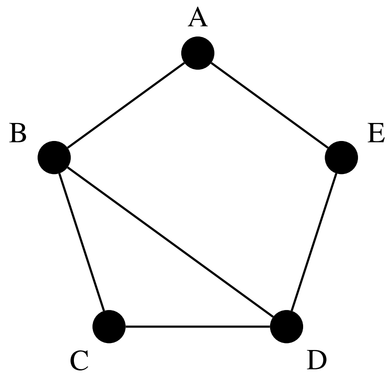


Figure 3.1: Labelled graph G_1

The graph G_1 in Figure 3.1 has $V(G_1)=\{a,b,c,d,e\}$ and $E(G_1)=\{ab,bc,cd,de,ea,bd\}$.

Definition 3.1.2 — Simple graph. A *simple graph* is a graph without loops, multiple edges, and directions.

R When we say *graph*, we mean a *simple graph* unless specified otherwise.

A graph with loops (e.g., some edge $\{v_1, v_1\}$) or multiple edges is called a multiple graph. In a directed graph, we call the lines arcs.

Definition 3.1.3
- Two vertices are *adjacent* or *neighbors* or *connected* if there exists an edge containing those two vertices.
- An edge is *incident* to a vertex if it contains that vertex.

Consider Figure 3.1. Then a and b are adjacent because $ab \in E$, a and c are not adjacent because $ac \notin E$, and the edge ab is incident to vertices a and b.

3.1.1 Subgraph

Definition 3.1.4 — Subgraph. Let $G = (V,E)$ and $H = (V',E')$ be graphs. Then H is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

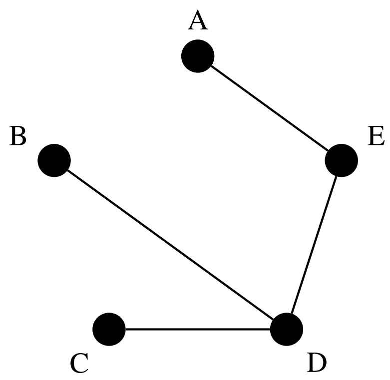


Figure 3.2: Subgraph G_1' of labelled graph G_1

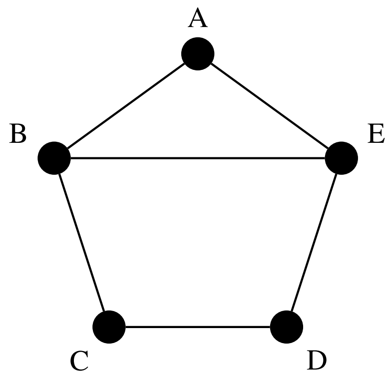


Figure 3.3: Non-subgraph G_1'' of labelled graph G_1

When looking at labelled graphs, we have to make sure the vertices and edges actually exist.

In an unlabelled graph, we have to show we can provide some labelling that makes the vertices and edges a subset.

3.2 Paths, cycles, and complete graphs

Definition 3.2.1
- A *path* is a graph P_n such that $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid 1 \le i \le n-1\}$.
- A *cycle* is a graph C_n such that $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{v_1 v_n\}$.
- A *complete graph* is a graph K_n such that $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $E(K_n) = \{v_i v_j \mid 1 \le i \ne j \le n\}$.

We will consider paths with $n \ge 2$, cycles with $n \ge 3$, and complete graphs with $n \ge 2$.

3.2.1 Connected graphs

Definition 3.2.2 — Connected graph. A graph is *connected* if there is a path between every pair of vertices.

A graph that is not connected is a *disconnected* graph

Definition 3.2.3 — Component. Let G be a graph. A maximal connected subgraph of G is called a *component* of G.

3.2.2 Degree

Definition 3.2.4 Let $G = (V, E)$ be a graph.
- The *degree* of a vertex v, denoted by $\deg(v)$ (or d_v or $d(v)$) is the number of edges incident to v.
- A vertex of degree zero is called an *isolated vertex*.
- The *minimum degree* in G is denoted by $\delta(G)$
- The *maximum degree* in G is denoted by $\Delta(G)$
- The *degree sequence* of G is a list of the degrees of each vertex in V (usually in non-increasing or non-decreasing order).

■ **Example 3.1** Consider the graph G_1 in Figure 3.1.

Then $\deg(a) = \deg(c) = \deg(e) = 2$ and $\deg(b) = \deg(d) = 3$. So $\delta(G) = 2$ and $\Delta(G) = 3$, and G has degree sequence $(2,2,2,3,3)$. ■

Exercise 3.1
(i) Find lower/upper bounds on degree and number of edges.
(ii) Find a graph whose vertices all have different degree.
(iii) Find a relationship between $|E(G)|$ and degrees.

Theorem 3.2.1 For every graph G on n vertices with m edges, we have $0 \le m \le \binom{n}{2}$.

Proof. Since $m \in \mathbb{Z}_{\ge 0}$, there are a maximum of $\binom{n}{2}$ two-element subsets of an n-set. □

Theorem 3.2.2 For every vertex v in a graph G on n vertices, we have $0 \le \deg(v) \le n-1$.

Proof. Each vertex can be adjacent to at most $n-1$ other vertices. □

The following result motivates a theorem on the degree of vertices.

Exercise 3.2 Is there a graph with degree sequence $(0,1,2,3,4)$?

There isn't a graph with such a degree sequence because we can't have both degree 0 and $n-1$ in a graph with n vertices.

Theorem 3.2.3 For $n \geq 2$, any graph on n vertices has at least two vertices of the same degree.

Proof. First we prove the following claim.

CLAIM: A graph can't have both 0 and $n-1$ in its degree sequence.

Proof. Assume there is a vertex of degree 0 and another vertex of degree $n-1$.

Since there is a vertex of degree 0, the graph is disconnected.

Since there is a vertex of degree $n-1$, the graph is connected.

Contradiction. □

So every vertex degree is in the set $\{0, 1, 2 \ldots, n-2\}$ or $\{1, 2 \ldots, n-1\}$. Both sets have size $n-1$.

$\therefore$ By pigeonhole principle, there must be two vertices of the same degree. □

3.3 Isomorphic Graphs

For graphs, the geometry does not matter; only the connections matter.

Definition 3.3.1 Let G, H be graphs. G and H are *isomorphic*, $G \cong H$, if there is a bijection $\sigma : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\sigma(u)\sigma(v) \in E(H)$. That is, the bijection preserves adjacency and non-adjacency. We call σ an isomorphism.

■ **Example 3.2** The graphs C_3 and K_3 are isomorphic, while the graphs C_3 and P_3 are not isomorphic. ■

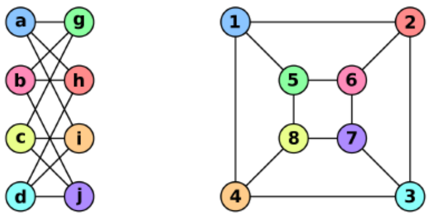


Figure 3.4: Isomorphic Graphs

Let G be the graph on the left and H the graph on the right. To show that $G \cong H$, we provide an isomorphism σ.

To prove two graphs are not isomorphic, we can show they differ in some structural property.

Some (necessary, but not sufficient) properties we can check are:

- Number of vertices
- Number of edges
- Number of components (i.e., connected, disconnected)
- Degree sequence
- Minimum degree
- Maximum degree
- Cycle structure (e.g., number of 3-cycles)
- Planar
- Bipartite
- Eigenvalues of adjacency matrices
- Chromatic number

We can also check this list for the complement of the graph.

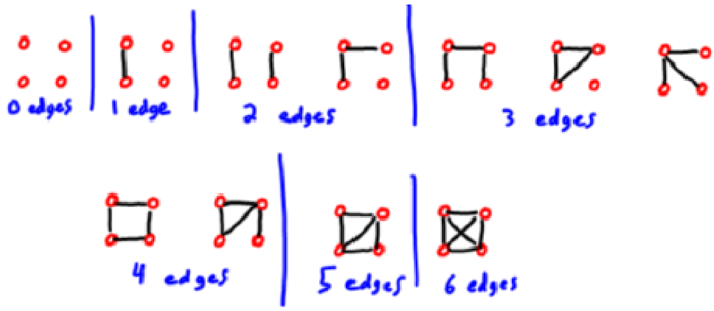


Figure 3.5: Non-isomorphic graphs of 4 vertices

3.4 Graph complement

Definition 3.4.1 — Complement. Let G be a graph. The complement of G, $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and edge set

$$E(\overline{G}) = \{xy \mid xy \notin E(G)\}$$

That is, we flip the edges and non-edges to draw the complement.

Theorem 3.4.1 Let G be a graph. Then $E(G) + E(\overline{G}) = \binom{n}{2}$

This is because we get all 2-element subsets of V.

Theorem 3.4.2 Let G and H be graphs. Then $G \cong H$ iff $\overline{G} \cong \overline{H}$.

We use the fact that if σ is an isomorphism for G and H, then we can use the same isomorphism for the complement, since it'll map non-edges to non-edges.

Exercise 3.3 How many (non-isomorphic) graphs are there on 4 vertices?

Exercise 3.4 How many edges does P_n have?
How many edges does C_n have?
How many edges does K_n have?

Proof. We have that $|E(P_n) = n - 1$, $|E(C_n)| = n$, and $|E(K_n)| = \binom{n}{2}$. □

3.5 Graph degree results

Theorem 3.5.1 — Handshaking Lemma. Let G be a graph. Then

$$\sum_{v \in V(G)} \deg(v) = 2\,|E(G)|\,.$$

Proof. We use a double counting argument (as Euler did).
PROBLEM: Count the ordered pairs $(v,e), v \in V(G), e \in E(G)$ where e is incident to v.
SOLUTION 1: Fix v. For each v, there are $\deg(v)$ such pairs.
Summing over v gives $\sum_{v \in V(G)} \deg(v)$ pairs.
SOLUTION 2: Fix e. For each e, there are 2 such pairs, one for each vertex incident to e.
Summing over e gives $\sum_{e \in E(G)} 2 = 2\,|E(G)|$.

Since both ways count the same number of pairs, they must be equal,

$$\sum_{v\in V(G)} \deg(v) = 2|E(G)|.$$

□

Corollary 3.5.2 A graph has an even number of vertices of odd degree.

Proof. Assume a graph has an odd number of vertices of odd degree. Then the total degree of the graph is odd, i.e, $\sum_{v\in V(G)} \deg(v) = 2k+1$ for some $k \in \mathbb{N}$.

Then by handshaking lemma, the number of edges in G is $\frac{\sum_{v\in V(G)} \deg(v)}{2} = \frac{2k+1}{2} = k+0.5 \notin \mathbb{N}$. Contradiction. □

Exercise 3.5 Let G be a graph with 31 edges and every vertex having degree at least 4. That is, $|E(G)| = 31$ and $\delta(G) \geq 4$.

(a) What is the maximum number of vertices that G can have?
(b) What is the minimum number of vertices that G can have?

Proof.
(a) We have by handshaking lemma $\sum_{v\in V(G)} \deg(v) = 2|E(G)| = 62$. So $\sum_{v\in V(G)} \deg(v) \geq \sum_{v\in V(G)} = 4|V(G)|$. Then $|V(G)| \leq 15$.
We have to show that this is the maximum. Consider $3K_5$ with one more edge. This shows 15 is possible.
(b) By Lemma 3.2.1, $31 \leq \binom{n}{2}$, so 9 is a lower bound (allowing 36 possible edges).
Consider the complement of P_6 in K_9, i.e., take the complete graph on 9 edges, then delete a path of 6 vertices. This gives 31 edges.

□

Theorem 3.5.3 Let G be a graph such that $\delta(G) \geq 2$. Then G contains a cycle.

G containing a cycle means there is a subgraph of G that is isomorphic to some C_n.

The following proof demonstrates the extremal principle, i.e., pick an object which is maximum/minimum among a specific class of structures; deduce there is an even larger/smaller object; give a contradiction.

Proof. Let $P = v_1v_2\cdots v_k$ be a longest path.

Consider v_1. Since $\deg(v_1) \geq 2$, v_1 is adjacent to another vertex w such that $w \neq v_1, v_2$.

Case 1: Suppose $w \notin V(P)$.

But then $wv_1v_2\cdots v_k$ is a path longer than P. Contradiction.

Case 2: Suppose $w \in V(P)$.

Then $w = v_i$ for some $i \neq 1,2$. This gives a cycle $C = v_1v_2\cdots v_iv_1$ in G. □

3.6 Bipartite Graphs

Definition 3.6.1 — Bipartite graph. A *bipartite graph* is a graph whose vertex set can be partitioned into two disjoint sets V_1, V_2 such that every edge has one endpoint in V_1 and the other endpoint in V_2.

Recall that a partition of a set V is a set of non-empty subsets of V such that every element $v \in V$ is in exactly one of these subsets. Therefore, $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$.

Definition 3.6.2 — Complete bipartite graph. A *complete bipartite graph* is a bipartite graph where every vertex of the first set V_1 is connected to every vertex of the second set V_2.

We use the notation $K_{m,n}$ where $m = |V_1|$ and $n = |V_2|$.

■ **Example 3.3** Which of the paths, cycles, and complete graphs are bipartite?

- For $n \geq 2$, P_n is bipartite.
- For $n \geq 3$, C_n is bipartite if and only if n is even.
- For $n \geq 3$, K_n is not bipartite.

Note that P_2, P_3, and C_4 are complete bipartite (they are $K_{1,1}$, $K_{1,2}$, and $K_{2,2}$ respectively). ■

Theorem 3.6.1 If a graph G is bipartite, then any subgraph H of G is bipartite.

Proof. Let $G = (V,E)$ be bipartite with vertex partition $V = X \cup Y$.

Let $H = (V',E')$ be a subgraph of G.

Choose $X' = X \cap V'$ and $Y' = Y \cap V'$. Then $V' = X' \cup Y'$ is a bipartition of H (otherwise, there is an edge of H with both endpoints in X' or Y' and thus this is also an edge of G with both endpoints in X or Y, which would contradict that $X \cup Y$ is a biparition of G). □

Theorem 3.6.2 If $G = (V,E)$ is a bipartite graph with bipartition (X,Y) (i.e., $V = X \cup Y$), then

$$|E(G)| = \sum x \in X \deg(x) = \sum y \in Y \deg(y)$$

The statement follows from the fact that every edge has exactly one endpoint in X and exactly one endpoint in Y.

Theorem 3.6.3 A graph G is bipartite if and only if G does not contain an odd cycle.

Proof. Let G be a bipartite graph. Then $V(G) = X \cup Y$ with $X \cap Y = \varnothing$ and every edge has one endpoint in X.

For contradiction, suppose G has an odd cycle.

WLOG, suppose $v_1 \in X$.

Then $v_2 \in Y$ since $v_1 v_2$ is an edge so has one endpoint in X and one endpoint in Y. So $v_3 \in X$, $v_4 \in Y$, $v_5 \in X$, ...

That is,

$$v_i \in \begin{cases} X, & \text{if } i \text{ is odd} \\ Y, & \text{if } i \text{ is even} \end{cases}$$

Then $v_k \in X$ since k is odd. But $v_k v_i$ is an edge with $v_1, v_k \in X$. Contradiction.

$\therefore$ G has no odd cycles.

Let G be a graph with no odd cycles (i.e., all cycles in G have even length).

Suppose G is connected. (If G is disconnected, we apply the following partition to each component.)

Fix a vertex $v \in V(G)$.

Define the distance between v and u, $\text{dist}(v,u)$, to be the number of edges in a shortest parth connecting them.

Then choose $X = \{x \mid \text{dist}(v,x) \text{ is even}\}$ and $Y = \{y \mid \text{dist}(v,y) \text{ is odd}\}$.

Since G is connected, $X \cup Y = V(G)$ since $\text{dist}(v,u)$ is either even or odd. Also, $X \cap Y = \varnothing$.

Now we need to show all edges of G have one endpoint in X and the other endpoint in Y.

For contradiction, suppose $x_1 x_2 \in E(G)$ with $x_1, x_2 \in X$. (We get an identical argument for $y_1, y_2 \in Y$.)

Let P be a shortest path from v to x_1, and let Q be a shortest path from v to x_2.

Then the length of P and Q have the same parity since $\text{dist}(v,x_1)$ and $\text{dist}(v,x_2)$ are even by the construction of X.

Let w be the last common vertex of P and Q (i.e., the path from w to x_1 and the path from w to x_2 have no common edges).

Let P_1 be the part of P from v to w, P_2 be the part of P from w to x_1, Q_1 be the part of Q from v to w, and Q_2 be part of Q from w to x_2.

Then P_1 and Q_1 have the same length (otherwise, say P_1 shorter than Q_1, then the path P_1Q_2 is shorter than Q from v to x_2, contradicting Q shortest.)

So P_2, Q_2 have the same parity (since P, Q have the same parity). So then the cycle from w to x_1 (by P_2) to x_2 (by supposition) to w (by Q_2) is an odd cycle. Contradiction.

$\therefore$ G is bipartite. □

3.7 Planar Graphs

■ **Example 3.4** The three utilities puzzle asks the following:

Three houses each need to be connected to gas, water and electric companies. Without using a third dimension or sending any connections through another house or company, is there a way to make all nine connections without crossings?

In graph theoretical terms:

Can the graph $K_{3,3}$ be drawn in the plane without any pair of edges crossing? ■

Definition 3.7.1 — Planar graph. A graph G is *planar* if it can be drawn in the plane so that no two edges intersect (except possibly at their edgepoints). Such a graph is called a *plane graph* or a *planar embedding* of G.

■ **Example 3.5** K_4 is planar. $K_{2,3}$ is planar. In fact, $K_{2,n}$ for any $n \in \mathbb{N}$ is planar. ■

Is K_5 or $K_{3,3}$ planar?

3.7.1 Faces

Definition 3.7.2 — Face. A plane graph G divides the plane into regions called *faces*, $F(G)$. Every plane graph has an unbounded region called the *exterior face*.

It is possible to have two different plane graphs which are isomorphic as graphs (where different means some face has a different number of boundary edges).

Is there a planar embedding of a graph G with a different number of faces?

Theorem 3.7.1 — Euler's formula. Let G be a connected plane graph such that $v = |V(G)|$, $e = |E(G)|$, $f = |F(G)|$. Then $v - e + f = 2$.

In general, $v - e + f = 1 +$ number of components.

Proof. Let G be a connected plane graph such that $v = |V(G)|$, $e = |E(G)|$, $f =$ number of faces of G.

We do induction on e.

BASE CASE: Let $e = 0$. Then $G = K_1$. (G is disconnected if $v \geq 2$.)

So we get $v = 1$, $e = 0$, $f = 1$.

$\therefore v - e + f = 1 - 0 + 1 = 2.$

INDUCTION HYPOTHESIS: Assume $v - e + f = 2$ for all connected plane graphs with less than e edges.

Let xy be an edge of G.
We have two cases.
(i) xy is on a cycle. So xy is a boundary edge of two faces.
Deleting xy merges the two faces (resulting in one face).
Call the new graph G'. It is still a connected plane graph.
Note that G' has v vertices, $e-1$ edges, and $f-1$ faces.
By IH, $v-(e-1)+(f-1)=2$.
$\therefore v-e+f=2$.
(ii) xy is not a cycle. So xy is the boundary edge of only one face.
Deleting xy does not change the number of faces, but disconnects the graph.
We get two subgraphs G_1 with v_1 vertices, e_1 edges, f_1 faces, and G_2 with v_2 vertices, e_2 edges, f_2 faces.
We get that $v = v_1 + v_2$, ...
By IH, since $e_1, e_2 < e$, we have that v_1...

□

Corollary 3.7.2 Every planar embedding of a planar graph has the same number of faces.

Proof. Let G be a planar graph with v vertices and e edges.

Let G_1 and G_2 be plane graphs that are both planar representations of G and suppose G_1 with v_1 vertices, e_1 edges, f_1 faces, and G_2 with v_2 vertices, e_2 edges, f_2 faces.

Since G_1 and G_2 are both drawings of G, they are isomorphic as graphs, implying $v_1 = v_2 = v$ and $e_1 = e_2 = e$.

By Euler's formula, $f_1 = 2+e_1-v_1 = 2+e-v$ and $f_2 = 2+e_2-v_2 = 2+e-v$.

$\therefore f_1 = f_2$. □

Definition 3.7.3 — Degree of a face. The degree of a face F is the length of its boundary, $\deg(F)$.

Think of each edge as having two sides. Edges that are entirely in one face (i.e., don't belong to any cycle) are counted twice to the degree of the face.

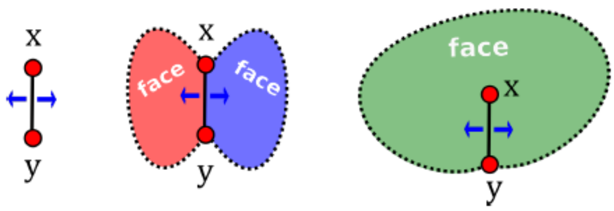


Figure 3.6: Counting edges for the degree of a face

Recall the handshaking lemma: Let G be a graph. Then $\sum_{v\in V(G)} \deg(v) = 2|E(G)|$. We get an analogous lemma for faces.

3.7.2 Handshaking lemma for faces

Theorem 3.7.3 — Handshaking lemma for faces. Let G be a plane graph. Then

$$\sum_{\text{face } F} \deg(F) = 2|E(G)|.$$

This holds because every edge either forms part of the boundary of two faces or appears twice on the boundary of a single face. (See Figure 3.6.) So either way, that edge contributes 2 to the total sum of degrees of the faces.

Corollary 3.7.4 — Corollary to Euler's formula. Let G be a connected planar graph with $v \geq 3$ vertices and e edges. Then $e \leq 3v - 6$.

Furthermore, if G is bipartite, then $e \leq 2v - 4$.

Proof. Suppose a graph G has a planar embedding with f faces.

Since G is connected and has at least 3 vertices, every face must have degree at least 3.

By handshaking lemma for faces, we have $2e = \sum_{\text{face } F} \deg(F) \geq \sum_{\text{face } F} 3 = 3f$.

So $f \leq \frac{2}{3}e$.

By Euler's formula, $v - e + f = 2$.

$$f = 2 - v + e \leq \frac{2}{3}e$$
$$\frac{1}{3}e \leq v - 2$$
$$e \leq 3v - 6$$

For a bipartite graph G, G has no odd cycles, so every face must have at least degree 4.

So $2e \geq 4f$, and by a similar argument with Euler's formula, $e \leq 2v - 4$. □

3.7.3 Nonplanar graphs

This lets us show the following graphs are not planar.

Theorem 3.7.5 K_5 is not planar.

Proof. Assume K_5 is planar.

Then $e \leq 3v - 6$ by the corollary to Euler's formula.

But $v = 5$ and $e = 10$, a contradiction since $3v - 6 = 9 < 10 = e$. □

Theorem 3.7.6 $K_{3,3}$ is not planar.

Proof. Assume $K_{3,3}$ is planar.

Then $e \leq 2v - 4$ by the corollary to Euler's formula since $K_{3,3}$ is bipartite.

But $v = 6$ and $e = 9$, a contradiction since $2v - 4 = 8 < 9 = e$. □

Definition 3.7.4 — Subdivision. An edge xy of a graph can be *subdivided* by placing a vertex somewhere along its length.

A graph which has been derived from G by a sequence of edge subdivision operations is called a *subdivision* of G.

Theorem 3.7.7 — Kuratowski's Theorem. A graph G is planar if and only if no subgraph of G is a subdivision of K_5 or $K_{3,3}$.

Proposition 3.7.8 Every subgraph of a planar graph is also planar (i.e., if a graph G contains a nonplanar subgraph, then G is not planar).

If we represent G as a plane graph, then its subgraphs are also plane graphs (by deleting edges and vertices, which does not produce edge crossing).

Proposition 3.7.9 Every subdivision of a planar graph is also planar (i.e., if a graph G is a subdivision of a nonplanar graph, then G is not planar).

Represent G as a plane graph. Then subdividing edges does not produce edge crossings, thus are also plane graphs.

These two propositions prove one direction of Kuratowski's theorem:

If G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.

■ **Example 3.6** The Petersen graph is nonplanar. Figure 3.7 shows that there is a subgraph which is a subdivision of $K_{3,3}$. ■

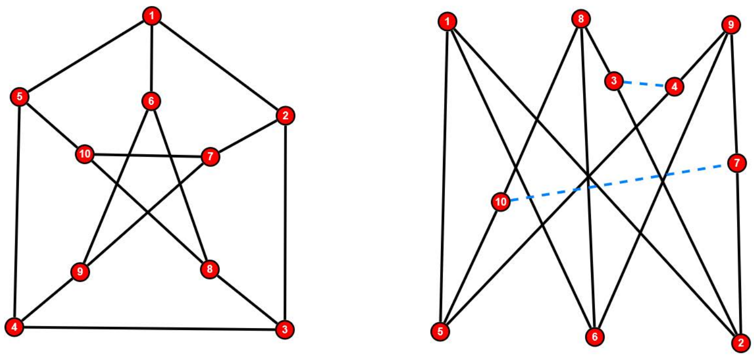


Figure 3.7

Exercise 3.6 Let G be a connected planar graph where every vertex has degree 3. If in a plane representation of G every face has degree either 5 or 6, and there are 20 faces of degree 6, then how many faces are there of degree 5?

3.8 Graph Colouring

Definition 3.8.1 — Graph colouring. A *colouring* of a graph G is an assignment of colours (or labels) to $V(G)$ so that adjacent vertices receive different colours.

Some textbooks use the phrase proper colouring to distinguish from colourings that don't respect adjacent vertices.

If k colours are used, we call the assignment a k-colouring.

■ **Example 3.7** Give a colouring of the Petersen graph. ■

Sometimes we use numbers instead of colours.

Definition 3.8.2 — k-colourable. A graph G is called *k-colouring* if G has a colouring with at most k colours.

Definition 3.8.3 — Chromatic number. The minimum k for which G is k-colourable (i.e., has a k-colouring) is called the *chromatic number* of G is denoted by $\chi(G)$.

Note that exhibiting a k-colouring gives an upper bound.

■ **Example 3.8** Compute the chromatic number for the paths, cycles, and complete graphs. ■

For $n \geq 2$, we have $\chi(P_n) = 2$.
For $n \geq 3$, we have

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

For $n \geq 2$, we have $\chi(K_n) = n$ because every vertex adjacent.

Exercise 3.7 Determine $\chi(H)$ given H in 3.8

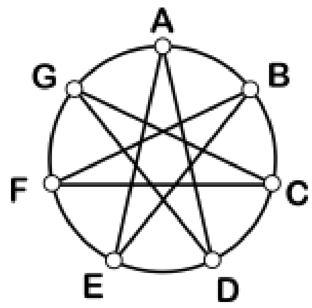


Figure 3.8: H

Proof. We prove $\chi(H) = 4$ by showing both $\chi(H) \geq 4$ and $\chi(H) \leq 4$.
We first show that $\chi(H) \geq 4$.
For contradiction, assume $\chi(H) \leq 3$ and suppose we have a colouring using colours $1,2,3$. Since vertices A, B, E form K_3, each has a different colour.
WLOG, suppose A has colour 1, B has colour 2, and C has colour 3.
Then D must have colour 2 (since D adjacent to A, E).
Then F must have colour 1 (since F adjacent to B, E).
Then G must have colour 3 (since G adjacent to D, F).
So C must have colour 3 (since C adjacent) □

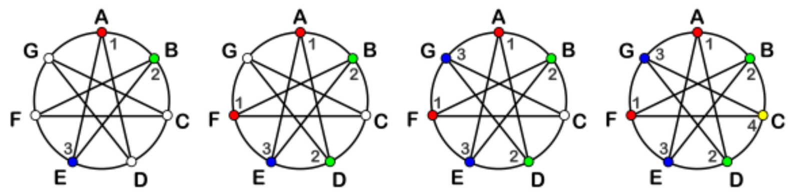


Figure 3.9: H

Theorem 3.8.1 For a graph G, $\chi(G) = 2$ if and only if G is a bipartite graph with at least one edge.

Note a graph with no edges is bipartite has $\chi(G) = 1$.

Proof. Suppose that $\chi(G) = 2$.
Then there is a 2-colouring of G; suppose it uses colours 1 and 2.
Let V_1 be the set of vertices of colour 1 and let V_2 be the set of vertices of colour 2.

Since we have a colouring, there are no edges whose endpoints are both in V_1 (otherwise adjacent vertices are both coloured with colour 1). Similarly, there are no edges whose endpoints are both V_2.

Therefore, the sets V_1 and V_2 form a bipartition of G, implying G is bipartite.

Finally, since $\chi(G) = 2$, we require 2 colours, thus G must have least one edge.

$\therefore$ G is a bipartite graph with at least one edge.

Suppose that G is a bipartite graph with at least one edge.

Let V_1 and V_2 form a bipartition of G.

Colour the vertices in V_1 with colour 1 and the vertices in V_2.

No pair of adjacent vertices have the same colour by the definition of bipartition.

Thus, this is a 2-colouring of G, implying $\chi(G) \leq 2$.

Since G has at least one edge, the endpoints of that edge must be different colours. So $\chi(G) \geq 2$.

$\therefore$ $\chi(G) = 2$. □

Definition 3.8.4 — Clique. A *clique* of a graph G is a complete subgraph.

The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G.

■ **Example 3.9** $\omega(K_{3,3}) = 2$ because $K_{3,3}$ bipartite, and thus has no odd cycles, so no $K_3 \cong C_3$.
■

3.8.1 Graph colouring bounds

The following theorem gives a lower bound on the chromatic number.

Theorem 3.8.2 Let G be a graph. Then $\chi(G) \geq \omega(G)$.

This follows because every vertex of a clique requires its own colour.

Many upper bounds are obtained from graph colouring algorithms.

Theorem 3.8.3 Let G be a graph on n vertices. Then $\chi(G) \leq n$.

We colour v_i by colour i, which produces an n-colouring since adjacent vertices must have different colours.

A better algorithm would be a greedy algorithm that uses "the least available colour".

- Let G be a graph on n vertices. Order the vertices as $v_1, v_2, \ldots, v_n$.
- Colour v_1 using colour 1.
- For $i = 2, 3, \ldots, n$, colour v_i the smallest colour that is not used on its lower-index neighbours.

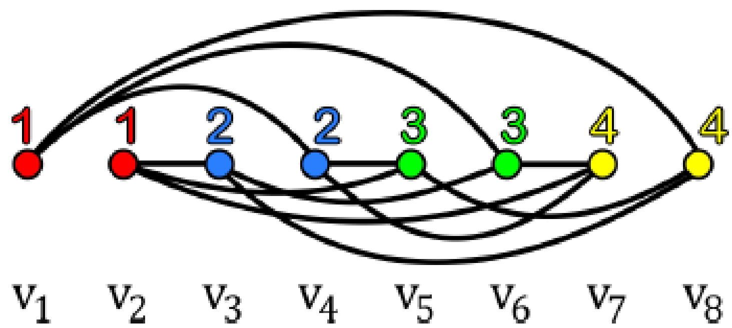


Figure 3.10

Note the graph in Figure 3.10 is actually bipartite.

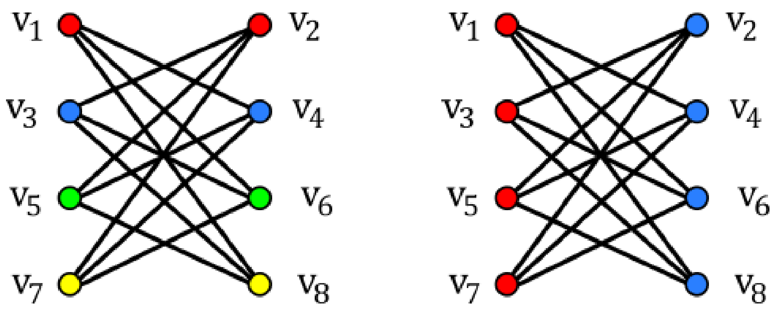


Figure 3.11

If we use the greedy algorithm to the vertex order $(v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_8)$, then we get the colouring on the right of Figure 3.11, which is optimal. So $\chi(G) = 2$.

Theorem 3.8.4 Let G be a graph with maximum degree $\Delta(G)$. Then $\chi(G) \leq \Delta(G) + 1$.

In the proof, we use a greedy colouring and note that in a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbours. So, one of $\{1, 2, \ldots, \Delta(G), \Delta(G)+1\}$.

We can reorder vertiecs in the greedy algorithm to give an optimal colouring because, by definition of $\chi(G)$, these is a colouring of G using $\chi(G)$ colours. Then you can put all the vertices of colour 1 in the beginning, then colour 2, colour 3, and so on.

Theorem 3.8.5 — Brooks' Theorem. If G is a connected that is not an odd cycle or complete graph, then $\chi(G) \leq \Delta(G)$.

3.9 Map colouring

Map-makers colour the different regions so that if two regions share a border, they are not coloured the same, making it easier to distinguish the border between them. In the past, using more colours increased the cost to produce the map, so we ask: Is there a bound on the number of colours required to colour any given map?

We can rephrase this problem in terms of planar graphs: How many colours are needed to colour a planar graph? If G is a planar graph, what is the best upper bound for $\chi(G)$?

In 1885, De Morgan sent a letter to Hamilton which made the following conjecture, the Four Colour Conjecture: Every planar graph is 4-colourable. In 1879, Kempe published a flawed proof, and in 1880, Tait also published a flawed proof. Heawood finds the flaw in Kempe's proof in 1890, then proves the Five Colour Theorem: Every planar graph is 5-colourable. Petersen finds the flaw in Tait's proof in 1891.

It isn't until 1976 that the Four Colour Theorem is proved by Appel and Haken (with assistance from Koch) notably using computers. They reduce the infinite number of possibilities ot a finite number, approximately 1936 configurations that were checked.

Theorem 3.9.1 — Lemma for Six Colour Theorem. Let G be a planar graph. Then G has a vertex of degree at most five.

Proof. If G has at most six vertices, the statement clearly holds, so assume G has at least seven vertices.

For contradiction, assume $\deg(v) \geq 6$ for all $v \in V(G)$.

By Handshaking lemma,

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq \sum_{v \in V(G)} 6 = 6|V(G)|.$$

Thus, $|E(G)| \geq 3|V(G)|$.

But since G is planar, by Corollary to Euler's formula, we must have $|E(G)| \leq 3|V(G)| - 6$. (Note we only prove this corollary for connected planar graphs, but it also holds for disconnected planar graphs.)

Thus $3|V(G)| \leq |E(G)| \leq 3|V(G)| - 6$, so $0 \leq -6$. Contradiction. □

Theorem 3.9.2 — Six Colour Theorem. Every planar graph G is 6-colourable, that is, $\chi(G) \leq 6$.

Proof. We use induction on the number of vertices in the graph.

BASE CASE: It holds for graphs with at most 6 vertices.

INDUCTION HYPOTHESIS: Assume it holds for planar graphs with less than n vertices.

INDUCTION STEP: Let G be a planar graph with n vertices. We want to show the statement holds for G.

By Lemma 3.9.1, G has a vertex v with $\deg(v) \leq 5$.

Delete vertex v and all incident edges to form the graph $G' = G - v$.

By IH, we can colour the vertices of G' with at most six colours. Since $\deg(v) \leq 5$, the neighbours of v use at most 5 colours.

So there is an unused colour that we can use to colour v which gives rise to a 6-colouring of G.

$\therefore \chi(G) \leq 6$ □

Theorem 3.9.3 — Five Colour Theorem. Every planar graph G is 5-colourable, that is, $\chi(G) \leq 5$.

This proof uses "Kempe" chains.

Proof. Let G be a smallest (least number of vertices) possible graph that is planar and requires 6 colours.

Let v be a vertex with $\deg(v) \leq 5$, which exists by Lemma 3.9.1.

We get the following cases:

(i) Consider when $\deg(v) \leq 4$.
We can delete v to form $G' = G - v$. Then G' can be coloured using ≤ 5 colours (otherwise G is not smallest planar graph requiring 6 colours).
Take a 5-colouring of G'. Now a colour is available for v from set $\{1,2,3,4,5\}$.
So G is 5-colourable. Contradiction.

(ii) Consider when $\deg(v) = 5$.
Again, we can delete v to form $G' = G - v$ and then G' can be coloured using ≤ 5 colours (otherwise G is not smallest planar graph requiring 6 colours).
If v has two neighbours which are coloured the same, then a colour is available for v, and so G is 5-colourable. Contradiction.
So consider instead when all 5 of v's neighbours have different colours.
Assume we have a plane graph of G. That is, we fix the position of v to be adjacent to v_i for each $i = 1,2,3,4,5$. WLOG, we colour v_i with i for each $i = 1$.
Consider as a subgraph H of G with all vertices with colours 1 and 3 and their edges.
   a) If v_1 and v_3 are in different components of H, then we can switch the colour classes 1 and 3 for one of the components, forming a new colouring of G' where v_1 has colour 3. Now colour 1 is available for v. So G is 5-colourable. Contradiction.

b) If v_1 and v_3 are in a single component of H, then it is possible that v_1 and v_3 are connected by an alternating path of colour $1,3,1,3,1,3$. So v is in this cycle from $C = vv_1Pv_3v$.
Consider v_2 and v_4 with colours 2 and 4 making a new subgraph H'. If they are in different components, we can do the same colour switch as earlier. If they are in the same component, we have a path P' in H connecting v_2 and v_4, creating a cycle $C' = vv_2P'v_4$. But this cycle must cross edges with C', so it is not planar. (These two cycles crossing is the Kempe chain.)

□

3.10 Eulerian Graphs

The Seven Bridges of Konigsberg is a problem Euler solved that led to the creation of graph theory, and it involves walks in multigraphs.

Definition 3.10.1 — Walk. A *walk* in a (multi-)graph G is a sequence

$$W = v_0e_1v_1e_2v_2\cdots v_{k-1}e_kv_k$$

whose terms alternate between vertices and edges (not necessarily distinct) such that $e_i = v_{i-1}v_i$ for $1 \le i \le k$. When G is a simple graph, we write $W = v_0v_1\cdots v_k$.

The *length of a walk* is the number of edges it contains.

Definition 3.10.2 — Types of walks. A *trail* is a walk such that all of its edges are distinct.

A *path* is walk such that all of its vertices and edges are distinct.

A *closed walk* is a walk whose initial and terminal vertices are the same (i.e., $v_0 = v_k$).

An *xy-walk* is a walk from vertex x to vertex y.

An *Euler trail* is a trail that visits every edge exactly once.

An *Euler tour* is a closed Euler trail.

An *Eulerian* graph is a graph with an Euler tour.

Theorem 3.10.1 Let G be a connected (multi-)graph. Then G is Eulerian if and only if every vertex of G has even degree.

Proof. Let G be Eulerian. Then it has an Euler tour. Tach passage of an Euler tour through a vertex uses two incident edges.

$\therefore$ Every vertex of G has even degree.

Suppose every vertex of G has even degree.

We use induction on the number of edges in the graph.

BASE CASE: True for 0, 1, 2, 3 edges. K_1 is trivially true. A two edge, two vertices multigraph, as well as K_3 have Euler tours.

INDUCTION HYPOTHESIS: Assume it holds for graphs with $\le m$ edges. (We're using strong induction.)

INDUCTION STEP: Let G be a connected graph with $m+1$ edges. We want to show the statement holds for G.

Suppose all vertices of G has even degree.

No vertex has degree 0 since it is connected.

All vertices x satisfy $\deg(x) \ge 2$. Then by Theorem 3.5.3 (cycle existence), G has a cycle C. Delete all edges of C.

Note that all vertices in G' has even degree (since vertices either lose 2 edges or none).

G' could be disconnected. By IH, since each component has $\leq m$ edges, they each have an Euler tour.

The following algorithm gives an Euler tour of G: traverse C but when a component of G' is entered for the first time, we detour along an Euler tour of that component.

□

Theorem 3.10.2 Let G be a connected (multi-)graph. Then G has an Euler trail if and only if G has at most two vertices of odd degree.

3.11 Independent Set

Definition 3.11.1 — Independent set. An *independent set* of a graph G is a set of vertices in which no two are adjacent.

Definition 3.11.2 — Maximal independent set. A *maximal independent set* is an independent set that is not a subset of any other independent set.

Definition 3.11.3 — Maximum independent set. A *maximum independent set* is a largest size independent set in a graph G.

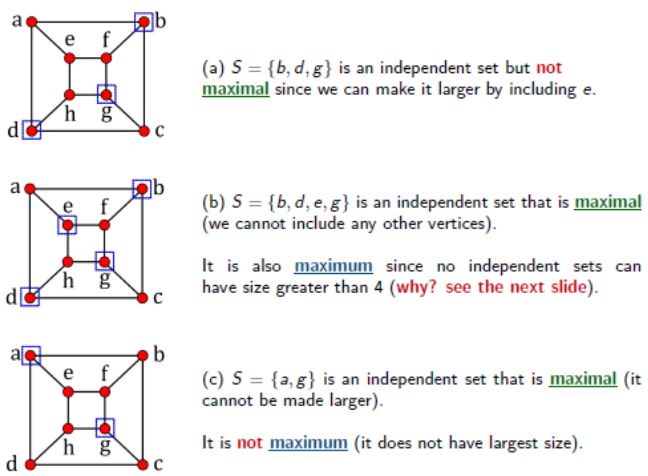


Figure 3.12

Note that the graph in Figure 3.12 is Q_3, the graph of a cube.

Definition 3.11.4 — Independence number. The size of a maximum independent set of a graph G is the *independence number* of G, $\alpha(G)$.

■ **Example 3.10** Consider the graph of a cube Q_3 (as shown in Figure 3.12).

$S = \{a, f, h, c\}$ is an independent set of size 4. $\therefore \alpha(G) \geq 4$.

For contradiction, suppose there exists S which is an independent set of S with size ≥ 5.

Consider edges ab, cd, ef, gh as pigeonholes, with vertices of S as pigeons. Then by pigeonhole principle, S contains one of these 2-element sets as a subset. Then S isn't independent. Contradiction.

$\therefore \alpha(G) \leq 4$.

$\therefore \alpha(G) = 4.$ ■

Theorem 3.11.1 Let G be a graph. Then $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

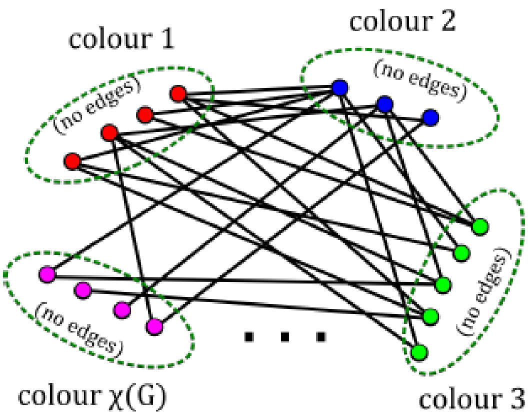


Figure 3.13

Proof. Consider a colouring of G using the minimum number $\chi(G)$ of colours.
Suppose the colours are $\{1, 2, \ldots, \chi(G)\}$.
Let S_i be the set of vertices of G with colour i. (See figure 3.13.)
Each set S_i is an independent set (by definition of a colouring).
By definition of independence number, $|S_i| \leq \alpha(G)$.
Then

$$|V(G)| = \sum_{i=1}^{\chi(G)} |S_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G) = \chi(G)\alpha(G)$$

$\therefore \chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. □

3.12 Hamilton cycles and paths

The icosian game was invented by William Hamilton in 1857 which involves tracing the edges of a dodecahedron such that each vertex is visited once and that you end where you start.

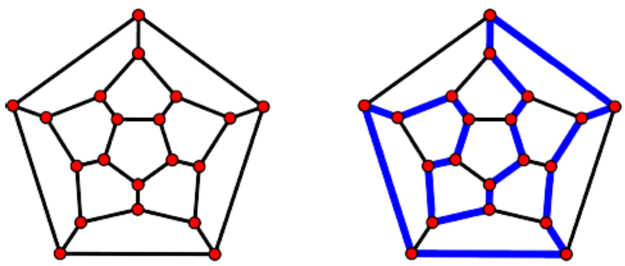


Figure 3.14

Definition 3.12.1 — Hamilton path. A path that uses every vertex of a graph G exactly once is a *Hamilton path*.

Definition 3.12.2 — Hamilton cycle. A cycle that uses every vertex of a graph G exactly once is a *Hamilton cycle*.

Definition 3.12.3 — Hamiltonian. A graph G that contains a Hamilton cycle is *Hamiltonian*.

A Hamilton cycle can be converted to a Hamilton path by deleting an edge.

■ **Example 3.11** The star graph $K_{1,n}$ for $n \geq 3$ (i.e., the complete bipartite graph with 1 vertex connected to n other vertices) has no Hamilton path or Hamilton cycle.

$K_5 \cup K_5$ with a single edge connecting the two graphs has a Hamilton path but not a Hamilton cycle. ■

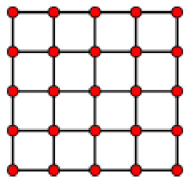


Figure 3.15

■ **Example 3.12** Consider the grid graph G in Figure 3.15.

It has a Hamilton path: start at the top-left, go right, go down one, go left, go down one, and repeat.

It does not have a Hamilton cycle: G is bipartite (consider a checkerboard pattern) with $|V_1| = 13$ and $|V_2| = 12$. ■

We can generalize to any $m \times n$ grid graph. The grid graph is Hamiltonian iff the product mn is even (so at least one of m, n is even).

Proposition 3.12.1 Let G be a bipartite graph with bipartition (V_1, V_2). If G has a Hamilton cycle, then $|V_1| = |V_2|$.

We can order the vertices in X and Y. So we can create a cycle $x_1 y_1 x_2 y_2 \ldots x_k y_l x_1$. We can't do this if the bipartition has different sized sets.

Consider the graph K_n with 1 extra vertex attached with one edge. It has $\binom{n-1}{2} + 1$ edges and is not Hamiltonian. So we can't just look at the number of edges.

Think about A3 Q5: if $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G) + 1$. This suggests a condition for a Hamilton cycle.

Theorem 3.12.2 — Dirac's Theorem. If a graph G has $n \geq 3$ vertices and $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Proof. Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq \frac{n}{2}$.

Let $P = v_1 v_2 \cdots v_k$ be a longest path.

Every neighbour of v_1 is on P (because otherwise, if neighbour w of v_1 not in P, then $P' = w v_1 \ldots v_k$ is longer than P, so P is not a longest path).

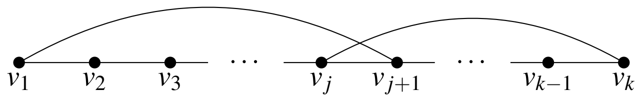


Figure 3.16: Hamilton Cycle C

Note that $k \geq \frac{n}{2}+1$ (see A3 Q5 argument: path must contain at least all the $\frac{n}{2}$ neighbours).

Consider when v_1 is adjacent to some v_{j+1} and v_k is adjacent to v_j (as in Figure 3.16). Then $C = v_1v_2 \dots v_jv_kv_{k-1}\dots v_{j+1}v_jv_1$.

CLAIM: There exists j such that $1 \leq j \leq k-1$ such that v_1 adjacent to v_{j+1} and v_k adjacent to v_j.

Proof. For contradiction, suppose for all edges v_1v_{j+1}, then there is no edge between v_kv_j. Then we get the following inequality:

$$\begin{aligned}|V(P)| &= |\{v_k\}| + |\{v_i \text{ adjacent to } v_k\}| + |\{v_i \text{ not adjacent to } v_k\}| \\ &\geq 1 + \deg(v_k) + \deg(v_1) \geq \\ &\geq 1 + \frac{n}{2} + \frac{n}{2} = n+1\end{aligned}$$

Contradiction. □

CLAIM: C is a Hamiltonian cycle.

Proof. For contradiction, suppose C is not a Hamiltonian cycle.

Then there is $w \notin V(P)$. Since $\deg(w) \geq \frac{n}{2}$ and $k \geq \frac{n}{2}+1$, there exists $v_i \in V(P)$ with w adjacent to v_i. Then there is a path $P' = wv_i\dots$ (which enters the cycle C at some vertex, then continues along the cycle) longer than P. □

□

3.13 Tree

Definition 3.13.1 — Tree. A *tree* is a connected graph with no cycles.

A disconnected graph with no cycles has trees as components, so we call such a graph a *forest*.

Definition 3.13.2 — Leaf. A vertex with degree 1 in a tree is a *leaf*.

Theorem 3.13.1 — Tree Theorem. Let G be a connected graph with $n \geq 2$ vertices. The following are equivalent:

(i) G is a tree.
(ii) G is acyclic. (There are no cycles in G.)
(iii) Every edge of G is a cut edge (i.e., deleting the edge disconnects G).
(iv) $|E(G)| = n-1$.

Proof of (i) ⇒ (iv). We prove this by induction on n.

BASE CASE: $n = 2$ holds.

INDUCTION HYPOTHESIS: Suppose $|E(G)| = n-1$ for trees with n vertices.

INDUCTION CASE: Let G be a tree with $n+1$ vertices. Note that if $\delta(G) \geq 2$, then G has a cycle. So by contrapositive, since G has no cycles, $\delta(G) \leq 1$. Since G is connected, $\delta(G) \geq 1$. So there exists a leaf x in G.

Consider $G' = G - x$. Then G' is a tree with n vertices. By IH, $|E(G')| = n-1$. Then $|E(G)| = |E(G')| + 1 = n = (n+1) - 1$. □

4. Counting Again

4.1 Principle of Inclusion-Exclusion

Say that we want to find the cardinality of a union of two sets. We count every thing in the first set, count everything in the second set, then subtract the overlap.

Theorem 4.1.1 If A, B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

When we have three sets, we have to consider the intersection of all three sets, and add that back. We can generalize this to n sets.

Theorem 4.1.2 — Principle of Inclusion-Exclusion (PIE). Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots A_n| = \left(\sum_{i=1}^{n} |A_i|\right) - \left(\sum_{1 \le i < j \le n} |A_i \cap A_j|\right) + \left(\sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|\right) + \cdots + ((-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|)$$

We can write this compactly as:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_k}|\right)$$

■ **Example 4.1** Let A, B be sets with $|A| = 4$ and $|9|$.
THen the maximum of $|A \cap B|$ can be 5, the minimum of $|A \cap B|$ can be 0.
Also $|A \cup B| + |A \cap B| = 4 + 9 - |A \cap B|$. Then $9 \le |A \cup B| \le 13$. ■

4.1.1 Complementary Forn of PIE

Proposition 4.1.3 Let S be a universal set with subsets A and B. We let $\overline{A}$ denote the complement of A in S.

Then $\left|\overline{A\cup B}\right| = |S| - |A\cup B| = |S| + (|A| + |B|) + |A\cap B|$.

Note that by DeMorgan's Law,

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}.$$

■ **Example 4.2** How many integers in $\{1,2,\ldots,100\}$ are not divisible by 2, 3 or 5? ■

We have to ensure we don't overcount the overlap such as with numbers divisible by both 2 and 3.

Proof. Let A_1 be the elements of S divisible by 2.

Let A_2 be the elements of S divisible by 3.

Let A_3 be the elements of S divisible by 5.

Then the answer to our problem is $\left|\overline{A_1}\cap\overline{A_2}\cap\overline{A_3}\right|$.

LEMMA: The number of positive integers divisible by k less than or equal to N is $\lfloor N/k \rfloor$.

By the lemma,

$$\begin{aligned}
|A_1| &= \lfloor 100/2 \rfloor = 50\\
|A_2| &= \lfloor 100/3 \rfloor = 33\\
|A_3| &= \lfloor 100/5 \rfloor = 20\\
|A_1\cap A_2| &= \lfloor 100/6 \rfloor = 16\\
|A_1\cap A_3| &= \lfloor 100/10 \rfloor = 10\\
|A_2\cap A_3| &= \lfloor 100/15 \rfloor = 6\\
|A_1\cap A_2\cap A_3| &= \lfloor 100/30 \rfloor = 4
\end{aligned}$$

Then, we can use PIE to calculate the number. □

■ **Example 4.3** Determine the number of integer solutions to $y_1+y_2+y_3+y_4 \le 70$ such that $1\le y_1\le 12$, $0\le y_2\le 10$, $-3\le y_3\le 13$, and $5\le y_4\le 35$. ■

We need this lemma:

Theorem 4.1.4 — Lemma for integer solutions. Let k_1 and $c_1,c_2,\ldots,c_k$ be integers. The number of integer solutions to $\sum_{i=1}^{k} x_i = n$ where $x_i \ge c_i$ for $i=1,\ldots,k$ is

$$\binom{\left(n-\sum_{i=1}^{k} c_i\right)+k-1}{k-1}.$$

A proof of this involves using the substitution $y_i = x_i - c_i \ge 0$, so that $\sum_{i=1}^{k} y_i = n - \sum_{i=1}^{k} c_i$

Proof. To get an equivalent problem, we add a slack variable y_5 with the equation $y_1+y_2+y_3+y_4+y_5 = 70$.

Then we substitute with new variables: $x_1 = y_1-1$, $x_2 = y_2$, $x_3 = y_3+3$, $x_4 = y_4-5$, and $x_5 = y_5$.

The following problem is equivalent:

Number of solutions to $x_1+x_2+x_3+x_4+x_5 = 67$ such that $0\le x_1\le 11$, $0\le x_2\le 10$, $0\le x_3\le 16$, $0\le x_4\le 30$, and $x_5\ge 0$.

Let S be the set of all nonnegative integer solutions to $x_1+x_2+x_3+x_4+x_5=67$. This will overcount due to upper bounds.

Let X_1 be the set of solutions where $x_1 \geq 12$ (and $x_i \geq 0$ for $i \neq 1$). Let X_2 be the set of solutions where $x_2 \geq 11$ (and $x_i \geq 0$ for $i \neq 2$). Let X_3 be the set of solutions where $x_3 \geq 17$ (and $x_i \geq 0$ for $i \neq 3$). Let X_4 be the set of solutions where $x_4 \geq 31$ (and $x_i \geq 0$ for $i \neq 4$).

Then we apply the lemma and use PIE.

$$
\begin{aligned}
|S| &= \binom{71}{4} \\
|X_1| &= \binom{59}{4} \\
|X_2| &= \binom{60}{4} \\
|X_3| &= \binom{54}{4} \\
|X_3| &= \binom{40}{4} \\
&\vdots
\end{aligned}
$$

□

■ **Example 4.4** Compute the number of arrangements of AAABBBCCC such that there are no three identical consecutive letters. ■

Proof. We know the total number of arrangemnets with no restrictions is $\binom{9}{3,3,3} = \frac{9!}{3!3!3!} = 1680$.

Let S be the set of all arrangements.

Let R_A be the subset of S of arrangements with 3 A's in a row.

Let R_B be the subset of S of arrangements with 3 B's in a row.

Let R_C be the subset of S of arrangements with 3 C's in a row.

Then use PIE. We want

$$\left|\overline{R_A} \cap \overline{R_B} \cap \overline{R_C}\right| = \left|\overline{R_A \cup R_B \cup R_C}\right| = S - (|R_A| + |R_B| + |R_C|) + \cdots$$

$|R_A| = \binom{7}{3,3,1}$. $|R_A \cap R_B| = \binom{5}{3,1,1}$. $|R_A \cap R_B \cap R_C| = \binom{3}{1,1,1}$.

□

4.2 Recurrence Relations

■ **Example 4.5** The Fibonacci sequence with pattern

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

can be defined using a recurrence:

$$F1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2} (n \geq 3).$$

A closed-form solution is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2, \ n \geq 1.$$

This is known as Binet's formula. ■

Definition 4.2.1 — Recurrence relation. Given a sequence $\{a_0, a_1, a_2, a_3, \ldots\}$ of numbers, a *recurrence relation* for a_n is an equation that relates the n-th term a_n to some of its predecessors in the sequence.

To initiate the computation, we require *initial conditions*.

The *solution* to a recurrence relation is an expression $a_n = f(n)$ where $f(n)$ is a function satisfying the recurrence and initial conditions.

■ **Example 4.6** Consider the recurrence relation

$$a_1 = 1, \ a_n = a_{n-1} + 1, \ (n \geq 2).$$

The pattern appears to be $a_n = n$.

We indeed have $a_1 = 1$. Also, the LHS is $a_n = n$ and the RHS is $a_{n-1} + 1 = (n-1) + 1$. ■

■ **Example 4.7** Consider the recurrence relation

$$a_1 = 1, \ a_n = na_{n-1}, \ (n \geq 2).$$

This is the definition for $n!$. ■

■ **Example 4.8** Consider the recurrence relation

$$a_0 = 1, \ a_n = \pi a_{n-1}, \ (n \geq 2).$$

This gives a geometric series $a_n = \pi^n$.

We indeed have $a_0 = \pi^0 = 1$. Also, the LHS is $a_n = \pi^n$ and the RHS is $\pi a_{n-1} = \pi \cdot \pi^{n-1} = \pi^n$. ■

See that the Fibonacci sequence also has a geometric solution.

■ **Example 4.9** Consider the recurrence relation

$$a_0 = 1, \ a_1 = 2, \ a_2 = 0 \qquad a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \ (n \geq 3).$$

■

This example is more complicated.

The first few terms are $\{1, 2, 0, 0, -4, -8, -20, -40, \ldots\}$. Ignoring initial conditions, we might guess a solution $a_n = 2^n$. Then LHS is $a_n = 2^n$ and RHS is $2(2^{n-1}) + 2^{n-2} - 2(2^{n-3}) = 2^n + 2^{n-2} - 2^{n-2=2^n}$. However, the initial conditions are not satisfied.

Observe that $a_n = c2^n$ for any constant c also solves the recurrence. Also observe that $a_n = c(-1)^n$ for any constant c also solves this recurrence. We can also observe that $a_n = c(1)^n = c$ for any constant c also solves this recurrence.

Are there any other solutions of the form x^n?

Assuming $a_n = x^n$ solves the recurrence, we get:

$$\begin{aligned} x^n &= 2x^{n-1} + x^{n-2} - 2x^{n-3} \\ x^n - 2x^{n-1} - x^{n-2} + 2x^{n-3} &= 0 \\ x^{n-3}(x^3 - 2x^2 - x + 2) &= 0 \\ x^{n-3}(x-1)(x+1)(x-2) &= 0 \end{aligned}$$

We get $x = 0$, $x = 1$, $x = -1$, $x = 2$. This shows that $(1)^n$, $(-1)^n$, and 2^n are solutions to the recurrence (ignoring 0^n).

We can take a linear combination of solutions to form new ones:

$$a_n = C_1(1)^n + C_2(-1)^n + C_3(2)^n.$$

All solutions satisfy the recurrence, but only some satisfy the initial conditions. Fix the initial conditions, then we get:

$$\begin{aligned} n = 0: && 1 &= C_1 + C_2 + C_3 \\ n = 1: && 2 &= C_1 - C_2 + 2C_3 \\ n = 2: && 0 &= C_1 + C_2 + 4C_3 \end{aligned}$$

Solving this system gives $C_1 = 2$, $C_2 = -2/3$, and $C_3 = -1/3$. So the solution to the original recurrence with the specified initial conditions is

$$a_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}(2)^n.$$

Definition 4.2.2 — Constant coefficient linear homogeneous recurrence relations. Let $\{a_n\}$ be a sequence. Then

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0 \quad (*)$$

where c_i is constant $(c_0, c_r \neq 0)$ and r is fixed $1 \leq r \leq n$ is an *rth order constant coefficient linear homogeneous recurrence relation*.

■ **Example 4.10**

- $a_n = a_{n-1} + a_{n-2}$ is 2nd order, constant coefficient, linear, and homogeneous.
- $a_n - 2a_{n-1} + 3a_{n-2} - a_{n-3}$ is 3nd order, constant coefficient, linear, and homogeneous.
- $a_n + a_{n-1} + (a_{n-2})^2$ is not linear.
- $a_n = a_{n-1} + a_{n-2} + 2$ is not homogeneous.
- $a_n = na_{n-1}$ is not constant coefficient.

■

Definition 4.2.3 — Characteristic equation. For $(*)$, replacing a_i with x_i and factoring out x^{n-r} gives

$$c_0 x^r + c_1 x^{r-1} + \cdots + c_{r-1} x + c_r = 0$$

which is called the *characteristic equation*. Its roots are called the *characteristic roots*.

Each characteristic root α gives a solution $a_n = \alpha^n$ of the recurrence.

Theorem 4.2.1 — Linear combinations of solutions. If $a_n = f(n)$ and $g(n)$ are solutions to $(*)$, then so is $a_n = C_1 f(n) = C_2 g(n)$.

Proof.

$$\begin{aligned} & c_0(C_1 f(n+C_2 g(n))) + c_1(C_1 f(n-1) + C_2 g(n-1)) + \cdots + c_r(C_1 f(n-r) + C_2 g(n-r)) \\ = & C_1(c_0 f(n) + c_1 f(n-1) + \cdots + c_r f(n-r)) + C_2(c_0 g(n) + c_1 g(n-1) + \cdots + c_r g(n-r)) \\ = & 0 + 0 = 0 \end{aligned}$$

□

Theorem 4.2.2 — Distinct Roots. If $\alpha_1, \alpha_2, \ldots, \alpha_r$ are distinct characteristic roots, then the general solution to $(*)$ is

$$a_n = C_1(\alpha_1)^n + C_2(\alpha_2)^n + \cdots + C_r(\alpha_r)^n$$

where $C_1, C_2, \ldots, C_r$ are constants dependant on the initial conditions.

Each $(\alpha_i)^n$ solves $(*)$, any linear combination of $(\alpha_i)^n$ solves $(*)$, so we get a set of r linearly independent solutions to $(*)$.

Theorem 4.2.3 — Repeated Roots. If $\alpha_1, \alpha_2, \ldots, \alpha_r$ are distinct characteristic roots with multiplicity m_i, then the general solution to $(*)$ is

$$\begin{aligned} a_n = & (C_{11} + C_{12} n + C_{13} n^2 + \cdots + c_{1m_1} n^{m_1 - 1})(\alpha_1)^n \\ & + (C_{21} + C_{22} n + C_{23} n^2 + \cdots + c_{1m_2} n^{m_2 - 1})(\alpha_2)^n \\ & + \cdots + (C_{k1} + C_{k2} n + C_{k3} n^2 + \cdots + c_{km_k} n^{m_k - 1})(\alpha_k)^n \end{aligned}$$

That is

$$a_n = P_1(n)(\alpha_1)^n + \cdots + P_k(n)(\alpha_k)^n$$

where $P_i(n)$ is a polynomial with degree less than m_i.

■ **Example 4.11** How many strings of length n are there using a's, b's, and c's such that no two a's are consecutive? ■

Proof. Let h_n be the number of allowed strings of length n.

The first letter can be a, b, c. Then we append the strings of length $n-1$ satisfying the condition.

The number of strings starting with b is h_{n-1}, and starting with c is h_{n-1}. If it's starting with a, then the next letters must be b or c. The number of strings starting with ab is h_{n-2} and starting with ac is h_{n-2}. This gives the recurrence

$$h_n = 2h_{n-1} + 2h_{n-2},\ n \geq 2.$$

We have initial conditions $h_0 = 1$ (include the empty string) and $h_1 = 3$. (We could start at h_1, h_2 instead, getting $h_2 = 8$.)

We get the characteristic equation by substituting $h_n = x^n$.

$$x^n = 2x^{n-1} + 2x^{n-2}$$
$$x^2 = 2x + 2$$

Then $x = 1 \pm \sqrt{3}$.
The general solution is $h_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$.
This gives the system of equations:

$$n = 0: \quad 1 = C_1 + C_2$$
$$n = 1: \quad 3 = C_1(1+\sqrt{3})^2 + C_2(1-\sqrt{3})^2$$

Solving the system, we get $C_1 = \frac{2+\sqrt{3}}{2\sqrt{3}}$ and $C_2 = \frac{-2+\sqrt{3}}{2\sqrt{3}}$. This gives the solution to the recurrence $h_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{-2+\sqrt{3}}{2\sqrt{3}}(1-\sqrt{3})^n$. The answer to the question is h_{10}. □

■ **Example 4.12** Solve the recurrence

$$a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4} \ (n \geq 4)$$

with $a_0 = 1$, $a_1 = 0$, $a_2 = 1$, $a_3 = 2$. ■

Proof. We get the characteristic equation

$$x^4 = -x^3 + 3x^2 + 5x + 2.$$

This can be factored into $(x+1)^3(x-2) = 0$. (Guess the roots, then do synethetic division until you have a quadratic.)

Then $x = 2, -1, -1, -1$ are the characteristic roots.

We need 4 linearly independent solutions. (If in doubt, then multiply by n to generate new solutions.)

So we get solutions $(-1)^n$, $n(-1)^n$, $n^2(-1)^n$, and 2^n. The general solution is

$$\left(C_1 + C_2 n + C_3 n^2\right)(-1)^n + C_4 2^n.$$

This gives the system of equations:

$$n = 0: \quad 1 = C_1 + C_4$$
$$n = 1: \quad 0 = (C_1 + C_2 + C_3)(-1) + 2C_4$$
$$n = 2: \quad 1 = (C_1 + 2C_2 + 4C_3) + 4C_4$$
$$n = 3: \quad 1 = (C_1 + 3C_2 + 9C_3)(-1) + 8C_4$$

Use row reduction to solve the system.

The constants are then $C_1 = 7/9$, $C_2 = -3/9$, $C_3 = 0$, and $C_4 = 2/9$.

$\therefore a_n = \left(\frac{7}{9} - \frac{3}{9}\right)(-1)^n + \frac{2^{n+1}}{9}$. □

■ **Example 4.13** Find a recurrence relation that has a solution $a_n = (-2)^n + (2+n)(1)^n$. ■

Proof. The numbers -2 and 1 are characteristic roots of multiplicities 1 and 2 (since $(1)^n$ has a degree one polynomial associated with it).

A possible characteristic equation is then $(x+3)(x-1)^2 = 0$.

So $x^3 = 3x - 2$ implying $3a_{n-2} - 2a_{n-3}$. Then we find the initial conditions by fixing n. □

Definition 4.2.4 — Constant coefficient linear non-homogeneous recurrence relations. Let $\{a_n\}$ be a sequence. Then

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = f(n) \quad (**)$$

where c_i is constant ($c_0, c_r \neq 0$) and r is fixed $1 \leq r \leq n$ is an *rth order constant coefficient linear non-homogeneous recurrence relation*.

First find a general solution $a_n^{(h)}$ for $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0$. Then find a paricular solution $a_n^{(p)}$ of $(**)$. The general solution is then $a_n = a_n^{(h)} + a_n^{(p)}$.

■ **Example 4.14** Solve $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$ with $a_0 = 3, a_1 = 8$. ■

Proof. First, solve the homogeneous recurrence.

Consider $a_n = 3a_{n-1} - 2a_{n-2}$. It has characteristic equation $x^2 = 3x - 2$, which has roots $x = 1$ and $x = 2$. So the general solution to the homogeneous recurrence is $a_n^{(h)} = A(1)^n + B(2)^n$.

Now find a particular solution for $a_n^{(p)}$. Guess and check with functions of the same form. $f(n) = 2^n$ suggests $a_n^{(p)} = C2^n$. Note this already is a solution to the homogeneous part, so multiply by n. Try $a_n^{(p)} = Cn2^n$.

Then we get the following equation:

$$Cn2^n = 3C(n-1)2^{n-1} - 2C(n-2)2^{n-2} + 2^n$$
$$4Cn = 6C(n-1) - 2C(n-2)$$

Solving for coefficients of n and 1, we get $4C = 6C - 2C$, and $0 = -6C + 4C + 4$, which has the solution $C = 2$.

That is, $c = 2$ solves the recurrence. So $a_n^{(p)} = 2n2^n = n2^{n+1}$.

Then $a_n = a_n^{(h)} + a_n^{(p)} = A(1)^n + B(2)^n + n2^{n+1}$.

To complete the solution, we solve for the initial conditions to give a system of equations. Then $A = 2$ and $B = 1$.

$\therefore a_n = 2 + 2^n + n2^{n+1}$

□

4.3 Generating Functions

A generating function encodes a sequence and allows us to solve combinatorial problems algebraically. Applications include finding exact formulas for the terms of a sequence, discovering new recurrence relations, proving combinatorial identities.

■ **Example 4.15** Determine the number of integer solutions to

$$a + b + c = n$$

where $0 \leq a \leq 2$, $0 \leq b \leq 1$, and $2 \leq c \leq 3$. ■

Proof. Consider the function $g(x) = \underbrace{(x^0 + x^1 + x^2)}_{a=0,1,2} \times \underbrace{(x^0 + x^1)}_{b=0,1} \times \underbrace{(x^2 + x^3)}_{c=2,3}$.

The answer to the problem is the coefficient of x^n in $g(x)$.

Consider expanding $g(x)$ and note the bijection between the ways to form x^n and solutions (a, b, c) to the problem.

For example, consider the term x^4.

$(x^0+x^1+x^2)$	(x^0+x^1)	(x^2+x^3)	a	b	c
x^0	x^1	x^3	0	1	3
x^1	x^0	x^3	1	0	3
x^1	x^1	x^2	1	1	2
x^2	x^0	x^2	2	0	2

When expanding, we get $g(x) = x^2+3x^3+4x^4+3x^5+x^6$. □

Definition 4.3.1 The coefficient of x^n in $g(x)$ is $[x^n]g(x)$.

We will need geometric series/sequences to find a closed form for $g(x)$.

Proposition 4.3.1 — Geometric series/sequences.

$$\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$$

and

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

We will use the following in particular:

■ **Example 4.16** (i) $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$
(ii) $1+x^2+x^4+x^6+\cdots = \frac{1}{1-x^2}$
(iii) $x^2+x^3+x^4+x^5+\cdots = x^2(1+x+x^2+x^3+\cdots) = \frac{x^2}{1-x}$ $(= \frac{1}{1-x}-1-x)$
(iv) $1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$ ■

■ **Example 4.17** Suppose we have several red, green, and blue balls. In how many ways can we select n balls if we must have at least two red, at most one green, and an even number of blue balls. ■

Proof. We want the number of integer solutions to $r+g+b=n$ with $r \geq 2$, $0 \leq g \leq 1$, and $b \leq 0$ is even.

Consider the function $g(x) = (x^1+x^2+x^3+\cdots)\times(x^0+x^1)\times(x^0+x^2+x^4+x^6+\cdots)$.

There is a bijection between combinations of red, green, blue and combinations of terms. For example, choosing 3 red, no green, and 4 blue corresponds to $x^3x^0x^4 = x^7$ in the expansion of $g(x)$.

Remark that $g(x)$ is a power series with an infinite number of terms.

We could choose 2 red balls OR 3 OR 4 OR $\cdots$, AND we could choose 0 green balls OR 1, AND we could choose 0 blue balls OR 2 OR 4 OR $\cdots$. Note how the $+$'s correspond to OR's and $\times$'s correspond to AND's.

The answer to the problem is $[x^n]g(x)$.

Note the derivative of Example 4.16(i) is $((1-x)^{-1})' = -1(1-x)^{-2}\cdot(-1)$, so we get $\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$ $(*)$.

Using Example 4.16, we get that

$$\begin{aligned} g(x) &= \left(\frac{x^2}{1-x}\right)(1+x)\left(\frac{1}{1-x^2}\right) \\ &= \left(\frac{x^2}{1-x}\right)(1+x)\left(\frac{1}{(1+x)(1-x)}\right) \\ &= x^2\frac{1}{(1-x)^2} \\ &= x^2(1+2x+3x^2+4x^3+\cdots) && \text{(By } (*)) \end{aligned}$$

(Use sigma notation.)
$\therefore g(x) = x^2+2x^3+3x^4+4x^5+\cdots = 0+0+\sum_{n=2}^{\infty}(n-1)x^n$

□

■ **Example 4.18** How many ways can we fill a box with n snacks if the number of chocolate bars is even, the number of cookies is a multiple of five, there are at most four pies, and there is at most one mooncake. ■

Proof. The generating function is $g(x) = (x^0+x^2+x^4+\cdots)\times(x^0+x^5+x^10+\cdots)\times(x^0+x^1+x^2+x^3+x^4)\times(x^0+x^1)$.

The answer to the problem is $[x^n]g(x)$. Using Example 4.16(i), (v), and the derivative of (i), we get

$$\begin{aligned} g(x) &= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)(1+x) \\ &= \frac{1}{(1-x)^2} \\ &= \sum_{n=0}^{\infty}(n+1)x^n \end{aligned}$$

$\therefore [x^n]g(x) = (n+1)$ for $n \geq 0$. □

Definition 4.3.2 — Generating function. Let $a_0, a_1, a_2, \ldots$ be a sequence. The *generating function* of the sequence is

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

■ **Example 4.19 — Some helpful power series.** (i) $\frac{a}{1-x} = \sum_{k=0}^{\infty} ax^k$ (Geometric series)

(ii) $\frac{1-x^{m+1}}{1-x} = \sum_{k=0}^{m} x^k$ (Geometric sequence)

(iii) $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, where $n \in \mathbb{Z}^+$ (Binomial theorem)

(iv) $(1-x^m)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^mk$, where $n \in \mathbb{Z}^+$ (substitute $-x^m$ into Binomial theorem)

(v) $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty}\binom{k+n-1}{k}x^k$ (special case of Generalized Binomial Theorem)

(vi) $\frac{1}{2}(e^x+e^{-x}) = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} = \cosh x$ and $\frac{1}{2}(e^x-e^{-x}) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} = \sinh x$

(vii) If $g(x) = \left(\sum_{i=0}^{\infty}a_ix^i\right)\left(\sum_{j=0}^{\infty}b_jx^j\right)$, then $[x^r]g(x) = \sum_{k=0}^{r}a_kb_{r-k}$. ■

■ **Example 4.20 — Generating functions for sequences.** (i) The generating function for the sequence $1, 1, 1, \ldots$ is $g(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

(ii) The generating function for the sequence $\binom{2022}{0}, \binom{2022}{1}, \binom{2022}{2}, \ldots, \binom{2022}{2022}, 0, 0, \ldots$ is $g(x) = \binom{2022}{0} + \binom{2022}{1}x + \binom{2022}{0}x^2 + \cdots + \binom{2022}{2022}x^{2022} = (1+x)^{2022}$

(iii) The generating function for the sequence $1, 2, 3, 4, 5, \ldots$ is $g(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$. ■

4.4 Generalized Binomial Theorem

Recall the binomial theorem (rewritten with $y = 1$).

Theorem 4.4.1 — Binomial theorem. For any integer $n \geq 0$, we have $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Isaac Newton ($\tilde{1}665$) generalized the binomial theorem to allow for n to to take on any real number (and in fact, it can be generalized to complex values of n). Instead of a finite sum, we get an infinite series. However, we must also generalize the notion of a binomial coefficient.

Definition 4.4.1 — Generalized binomial coefficient. For $a \in \mathbb{R}$ and $k \in \mathbb{Z}^+$, define

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} \qquad \left(= \frac{a}{k} \cdot \frac{(a-1)}{(k-1)} \cdots \frac{a-k+1}{1} \right).$$

Also set $\binom{a}{0} = 1$.

See we can rewrite $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$, so we're using this to generalize: start at a and count down k times.

■ **Example 4.21** (i) $\binom{-2}{5} = \frac{(-2)(-3)(-4)(-5)(-6)}{6!} = -6$

(ii) $\binom{1/3}{3} = \frac{(1/3)(1/3-1)(1/3-2)}{3!} = \frac{5}{81}$ ■

Theorem 4.4.2 — Generalized binomial theorem. For any nonzero real number $a \in \mathbb{R}$, we have $(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k$.

Theorem 4.4.3 If $n \in \mathbb{Z}^+$, then $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$.

Proof. By definition, we have

$$\begin{aligned}
\binom{-n}{k} &= \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} \\
&= \frac{(-1)^k n(n+1)\cdots(n+k-1)}{k!} \\
&= (-1)^k \binom{n+k-1}{k}
\end{aligned}$$

□

Corollary 4.4.4 If $n \in \mathbb{Z}^+$, then $\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k$.

■ **Example 4.22** Let $g(x) = \frac{1}{(1-x)^4}$ be a generating function. What is the coefficient of x^n in its expansion? That is, find $[x^n]\frac{1}{(1-x)^4}$. ■

Proof. Using Corollary 4.4.4, and substituting x with $-x$ and setting $n = 4$, we get

$$\begin{aligned}
\frac{1}{(1-x)^4} &= \sum_{k=0}^{\infty}(-1)^k\binom{3+k}{k}(-x)^k \\
&= \sum_{k=0}^{\infty}(-1)^2k\binom{3+k}{k}x^k \\
&= \sum_{k=0}^{\infty}\binom{3+k}{k}x^k
\end{aligned}$$

$\therefore [x^n]\frac{1}{(1-x)^4} = \binom{3+k}{k}$. □

■ **Example 4.23** Find the number of integer solutions to $x_1 + x_2 + x_3 = n$ where $x_1 \geq 0$, $0 \leq x_2 \leq 2$, $x_3 \geq 0$ and x_3 must be even. ■

Proof. The generating function is $g(x) = (1+x+x^2+\cdots)(1+x+x^2)(1+x^2+x^4+\cdots)$, and the number of integer solutions is equivalent to the coefficient $[x^n]g(x)$.

Simplifying gives $g(x) = \frac{1}{1-x}(1+x+x^2)\frac{1}{1-x^2} = \frac{1+x+x^2}{(1-x)(1-x)(1+x)} = \frac{1+x+x^2}{(1-x)(1-x)^2}$.

We can extract $[x^n]g(x)$ by using partial fractions: $\frac{1+x+x^2}{(1-x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$.

Then $1+x+x^2 = A(1-x)^2 + B(1+x)(1-x) + C(1+x)$.

We could equate the coefficients on both sides after expanding to get a system of equations. So the coefficient of x^2 is $1 = -A + C$, the coefficient of x is $1 = B - 2C$, and the coefficient of 1 is $1 = A + B + C$.

Alternatively, we could plug in values for x.

When $x = 1$, we get $3 = 2C$, so $C = 3/2$. When $x = -1$, we get $1 = 4A$, so $A = 1/4$. When $x = 0$, $1 = A + B + C$, so $B = -3/4$.

$$\begin{aligned}
g(x) &= \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} \\
&= \frac{1/4}{1+x} + \frac{-3/4}{1-x} + \frac{3/2}{(1-x)^2} \\
&= \frac{1}{4}\cdot\frac{1}{1+x} - \frac{3}{4}\cdot\frac{1}{1-x} + \frac{3}{2}\cdot\frac{1}{(1-x)^2} \\
&= \frac{1}{4}\sum_{k=0}^{\infty}(-x)^k - \frac{3}{4}\sum_{k=0}^{\infty}x^k + \frac{3}{2}\sum_{k=0}^{\infty}(-1)^k\binom{2+k-1}{k}(-x)^k && \text{(Corollary 4.4.4)} \\
&= \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k x^k - \frac{3}{4}\sum_{k=0}^{\infty}x^k + \frac{3}{2}\sum_{k=0}^{\infty}(k+1)x^k && \left(\binom{k+1}{k} = k+1\right) \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{4}(-1)^k - \frac{3}{4} + \frac{3}{2}(k+1)\right)x^k
\end{aligned}$$

$\therefore [x^n]g(x) = \frac{(-1)^n}{4} - \frac{3}{4} + \frac{3}{2}(n+1)$ □

■ **Example 4.24** How many ways can you tile a $2 \times n$ board completely using dominoes (i.e., tiles of size 1×2 and 2×1)? ■

Proof. Let a_n be the number of ways to tile the $2 \times n$ board with dominoes.

Then $a_1 = 1$, $a_2 = 2$, $a_3 = 3$.

When the board starts with a vertical domino on the bottom left, a $2 \times n-1$ board remains. When the board starts with a horizontal domino on the bottom left, then there has to be a horizontal domino on the top left square, giving a $2 \times n-2$ board to solve. So we get the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$, $a_1 = 2$.

Consider the generating function

$$\begin{aligned} g(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \end{aligned}$$

We can write these two series in terms of $g(x)$. In particular, $\sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3 + \cdots = x(-a_0 + a_0 + a_1 x + \cdots) = x(a_0 + g(x))$ and $\sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + \cdots = x^2 g(x)$.

Then $g(x) = 1 + x + x(g(x) - 1) + x^2 g(x)$, which gives a closed form for $g(x)$. Then we use partial fractions.

□

4.5 Ordinary Generating Functions

Theorem 4.5.1 Let $n, k \in \mathbb{Z}^+$ and $T_1, T_2, \ldots, T_k$ be sets of non-negative integers.

If a_n is the number of integer solutions satisfying $x_1 + x_2 + \cdots + x_k = n$ such that $x_1 \in T_n, \ldots, x_k \in T_k$, then a_n is equal to the coefficient of x^n in the expansion of the generating function

$$g(x) = \left(\sum_{t_1 \in T_1} x^{t_1} \right) \cdots \left(\sum_{t_k \in T_k} x^{t_k} \right) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

This is also equal to the number of ways to partition n identical objects into k labelled groups such that the number of objects in the ith group is an element of T_i.

When $T_i = \{0, 1, 2, \ldots\}$ for $1 \leq i \leq k$, we get our usual stars and bars result since the coefficient of x^n in the expansion of $g(x)$ is equal to $\binom{n+k-1}{n}$ by Generalized Binomial Theorem:

$$g(x) = (1 + x + x^2 + \cdots)^k = \frac{1}{(1-x)^k} = \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^i.$$

4.6 Exponential Generating Functions

Definition 4.6.1 — Exponential generating function. Let a_n be a sequence. The *ordinary* *generating function* of the sequence is

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

while its exponential generating function is

$$G(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}.$$

■ **Example 4.25** The sequence $1, 1, 1, \ldots, 1, \ldots$ has

- $g(x) = \sum_{k=0}^{\infty} x^K = \frac{1}{1-x}$
- $G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

■

■ **Example 4.26** The sequence $\left\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, \ldots\right\}$ has ordinary generating function

$$g(x) = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (1+x)^n.$$

■

■ **Example 4.27** Let $P(n,k) = \frac{n!}{(n-k)!}$ be the number of permutations of n objects taken k at a time. The exponential generating function for the sequence $\{P(n,0), P(n,1), \ldots, P(n,n), 0, 0, \ldots\}$ is

$$\begin{aligned} G(x) &= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + \cdots + P(n,k)\frac{x^k}{k!} + \cdots + P(n,n)\frac{x^n}{n!} \\ &= \cdots \\ &= (1+x)^n \end{aligned}$$

■

So $(1+x)^n$ is the ordinary generating function for a sequence of combinations while it's the exponential generating function for permutations, suggesting we should use ordinary or exponential depending on the respective problem.

Recall the definition of multinomial coefficient:

Definition 4.6.2 — Multinomial coefficient. Let n be a positive integer and $n_1, n_2, \ldots, n_k$ be non-negative integers such that

$$n_1 + n_1 + \cdots + n_k = n$$

The *multinomial coefficient* is defined as

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1! n_2! \cdots n_k!}$$

Theorem 4.6.1 If there are $n_i \geq 1$ objects of type i for $1 \leq i \leq k$, and there are $n = n_1 + n_2 + \cdots n_k$ objects in total, then the number of arrangements of these n objects is $\binom{n}{n_1, n_2, \ldots, n_k}$.

■ **Example 4.28** How many ways can 4 of the letters from PAPAYA be arranged? ■

Proof. There are three A's, two P's, one Y.

Consider the exponential generating function

$$G(x) = \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) \cdot \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!}\right) \cdot \left(\frac{x^0}{0!} + \frac{x^1}{1!}\right).$$

The answer to the problem is the coefficient of $\frac{x^4}{4!}$ in $G(x)$ because there is a bijection between the number of ways to form an $\frac{x^4}{4!}$ term and the number of arrangements of 4 elements from A, A, A, P, P, Y.

$$\begin{aligned} G(x) &= \cdots + \left(\frac{x^1}{1!}\frac{x^2}{2!}\frac{x^1}{1!} + \frac{x^2}{2!}\frac{x^1}{1!}\frac{x^1}{1!} + \frac{x^2}{2!}\frac{x^2}{2!}\frac{x^0}{0!} + \frac{x^3}{3!}\frac{x^1}{1!}\frac{x^0}{0!} + \frac{x^3}{3!}\frac{x^0}{0!}\frac{x^1}{1!}\right) + \cdots \\ &= \cdots + \left(\frac{4!}{1!2!1!} + \frac{4!}{2!1!1!} + \frac{4!}{2!2!0!} + \frac{4!}{3!1!0!} + \frac{4!}{3!0!1!}\right)\frac{x^4}{4!} \end{aligned}$$

□

Theorem 4.6.2 Let $n, k \in \mathbb{Z}^+$ and $T_1, T_2, \ldots, T_k$ be sets of non-negative integers.

Suppose we have k letters with an unlimited number of each type. Let A be our alphabet $A = \{A_1, A_2, \ldots, A_k\}$.

If a_n is the number of length n arrangements of letters from A such that the number of A_i's used is an integer in T_i, then a_n is the coefficient of $\frac{x^n}{n!}$ in the expansion of the exponential generating function

$$G(x) = \left(\sum_{t_1 \in T_1} \frac{x^{t_1}}{t_1!}\right) \cdots \left(\sum_{t_k \in T_k} \frac{x^{t_k}}{t_k!}\right) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!} + \cdots.$$

This is also equal to $a_n = \sum \binom{n}{n_1, n_2, \ldots, n_k}$ where the summation is over all non-negative solutions $(n_1, n_2, \ldots, n_k)$ of $n_1 + n_2 + \cdots + n_k = n$ such that $n_i \in T_i$ for $i = 1, 2, \ldots, k$.

■ **Example 4.29** How many strings of length n can be formed using A's, B's, and C's so that the number of A's is odd and the number of B's is also odd? ■

Proof. Use $\sinh x \cdot \sinh x \cdot e^x$. □

■ **Example 4.30** Let h_n be the number of ways to colour squares of a $1 \times n$ grid using red, green, and blue so that an even number of red is used. Determine a formula for h_n. ■

Proof. We could solve this with recurrence relations. We have $h_1 = 2$ and $h_2 = 5$. We set up a recurrence on the first square.

If it is green, there are h_{n-1} colourings.

If it is blue, there are h_{n-1} colourings.

If it is red, we need to colour a $1 \times n - 1$ grid with an odd number of red. This is equal to $3^{n-1} - h_{n-1}$, the total number of RGB colourings of a $1 \times n - 1$ grid subtracted by the number of colourings of a $1 \times n-1$ grid with an even number of red.

So we get that $h_n = 2h_{n-1} + (3^{n-1} - h_{n-1}) = h_{n-1} + 3^{n-1}$ with $h_1 = 2$.

While we have other methods for solving this recurrence, we could also iterate and find a pattern.

$$\begin{aligned}
h_1 &= 2 \\
h_2 &= h_1 + 3 && = 2+3 \\
h_3 &= h_2 + 3^2 && = 2+3+3^2 \\
h_4 &= h_3 + 3^3 && = 2+3+3^2+3^3 \\
&\vdots \\
h_n &= h_{n-1} + 3^{n-1} && = 2+3+3^2+\cdots+3^{n-1}
\end{aligned}$$

So $h_n = 2 + \sum_{n-1}^{k=0} 3^k = \cdots = \frac{3^n+1}{2}$.

We would have to prove this formula satisfies the recurrence and initial condition.

We can also find the coefficient of $\frac{x^n}{n!}$ for the exponential generating function.

$$G(x) = \left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \cdot \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right) \cdot \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)$$

We have that $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, $\frac{1}{2}(e^x - e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh x$, and $\frac{1}{2}(e^x + e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh x$.

Then we find the coefficient.

$$\begin{aligned}
G(x) &= \left(\frac{1}{2}(e^x + e^{-x})\right) \cdot e^x \cdot e^x = \frac{1}{2}\left(e^{3x} + e^x\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!} + \sum_{k=0}^{\infty} \frac{x^k}{k!}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} (3^k + 1)\frac{x^k}{k!}
\end{aligned}$$

Therefore, $h_n = \left[\frac{x^n}{n!}\right] G(x) = \frac{1}{2}(3^n + 1)$. □

■ **Example 4.31** Let h_n be the number of ways to colour squares of a $1 \times n$ grid using red, green, and blue so that an even number of red is used and at least one blue. Determine a formula for h_n. ■

Proof. We find the coefficient of $\frac{x^n}{n!}$ for the exponential generating function.

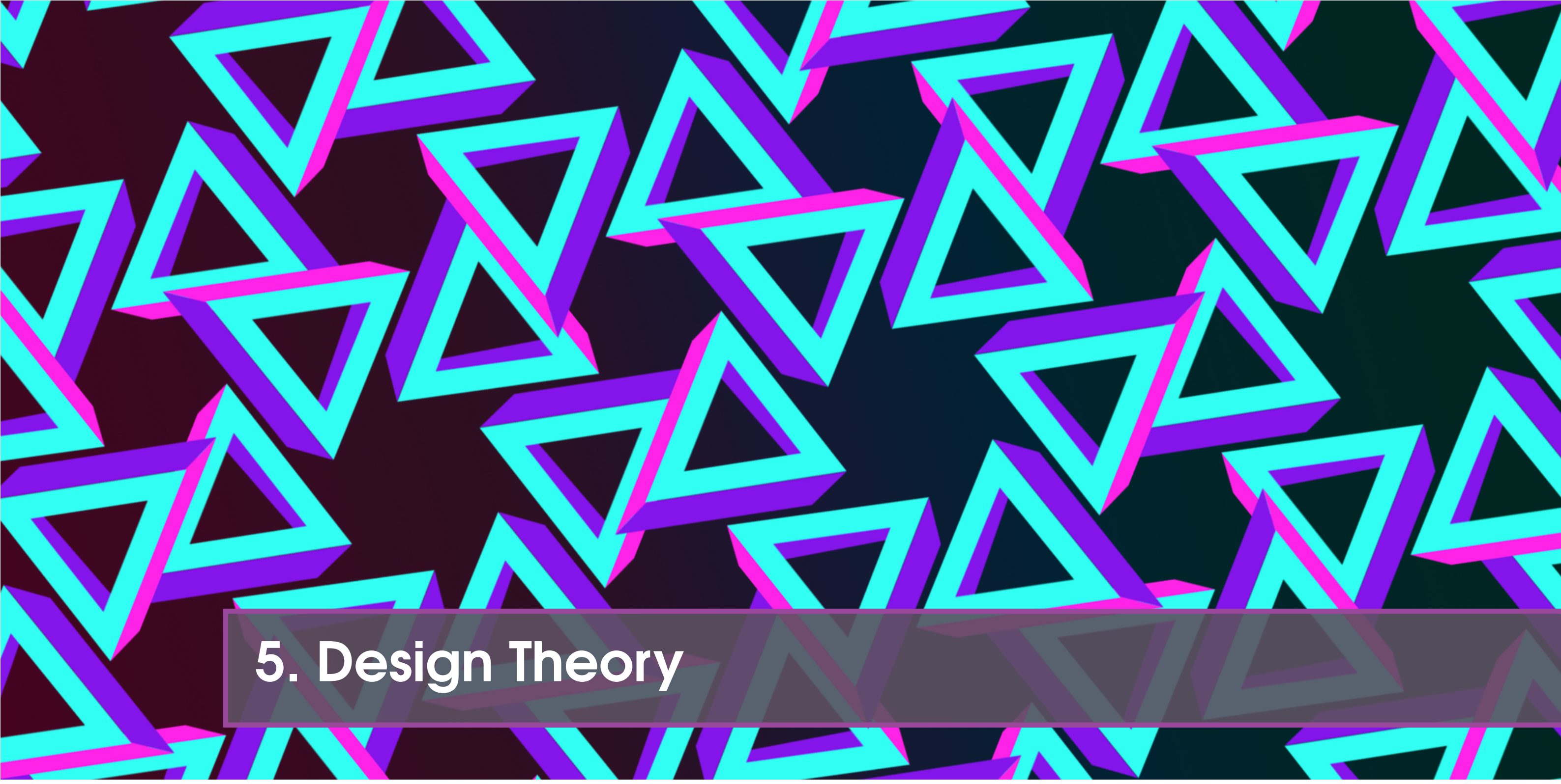
$$G(x) = \left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \cdot \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right) \cdot \left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

We have that $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, $\frac{1}{2}(e^x - e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh x$, and $\frac{1}{2}(e^x + e^{-x}) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh x$.

Then we find the coefficient.

$$\begin{aligned} G(x) &= \left(\frac{1}{2}(e^x + e^{-x})\right) \cdot e^x \cdot (e^x - 1) \\ &= \frac{1}{2}(e^{3x} - e^{2x} + e^x - 1) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}\right) - \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}\right) + \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) - \frac{1}{2} \\ &= -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{3^k - 2^k + 1}{2} \frac{x^k}{k!} \end{aligned}$$

Therefore, $h_0 = -\frac{1}{2} + \frac{3^0 - 2^0 + 1}{2} = 0$ and for $n \geq 1$, $h_n = \sum_{k=0}^{\infty} \frac{3^k - 2^k + 1}{2}$. □



5. Design Theory

5.1 Combinatorial Designs

Designs are a generalization of graphs. Instead of taking 2-element subsets, we could take 3-element subsets.

■ **Example 5.1** Let $X = \{0,1,2,3,4,5\}$ and $\mathcal{B} = \{012,023,034,045,051,124,235,341,452,513\}$.

We call the elements of X *points* (instead of vertices). $\mathcal{B}$ is a collection of 3-element subsets of X. We call the elements of $\mathcal{B}$ *blocks*, and $\mathcal{B}$ is the block set. ■

Consider any two pairs of points. For example, $0,1$ in $012,015$, $0,2$ in $012,023$, and $3,5$ in $235,513$. Every pair of points is in exactly 2 points.

How do we visualize (X,β)? We could draw a pentagon with a point insde (the centre point labelled by 0). Then blocks correspond to triples with exactly one pentagon edge.

Definition 5.1.1 — Design. Let t,k,v,λ be integers with $t < k < v$ and $\lambda > 0$.

A $t-(v,k,\lambda)$ *design* is a pair $(X,\mathcal{B})$ such that:

- X is a set of cardinality v whose elements are called *points*,
- $\mathcal{B}$ is a collection of k-subsets of X called *blocks*,
- and any t points are contained in exactly λ blocks.

The main problem is: *For which values of parameters do designs exist?*

In Example 5.1, we have a $2-(6,3,2)$ design.

Definition 5.1.2 — Steiner system. A t-design with $\lambda = 1$ is called a *Steiner system* denoted by $S(t,k,v)$.

That is, t points are contained in exactly 1 block.

$S(2,q+1,q^2+q+1)$ is called the *finite projective plane*.

A Steiner system $S(2,3,n)$ is a *Steiner triple system*.

Definition 5.1.3 — Steiner triple system. A *Steiner triple system*, denoted by $STS(n)$, consists of a set X of n points and a set $\mathcal{B}$ of 3-element subsets of X (called blocks or triples), with the property that any two points of X lie in a unique triple. We call n the *order* of the Steiner triple system.

Again, the main problem is: *For which values of n does a* $STS(n)$ *exist?*

■ **Example 5.2** $STS(3)$ is the design where $X = \{1,2,3\}$ and each pair $1,2$, $1,3$, and $2,3$ are in exactly one block. We get $\mathcal{B} = \{123\}$.

However, $STS(n)$ does not exist for $n = 4,5,6$.

WLOG, we have a block 123. This forces a second block with 14, but then there is no other pair we can add.

$STS(7)$ exists with $X = \{1,2,3,4,5,6,7\}$ and $\mathcal{B} = \{123,145,167,246,257,347,356\}$.

This is called the Fano plane, as shown in Figure 5.1. It is a projective plane of order $q = 2$. In a finite projective plane: for every pair of distinct points, there is exactly one line with both points; there is a set of four points such that no three belong to the same line; and the intersection of any two distinct lines contains exactly one point. ■

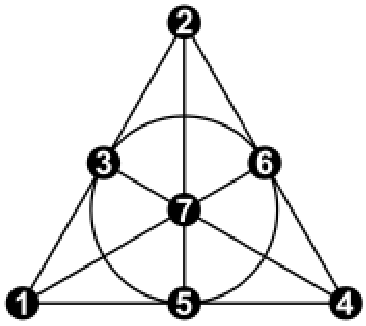


Figure 5.1: The Fano Plane

This was the original problem solved by Kirkman in 1847, which happened to be a Steiner system.

■ **Example 5.3 — Kirkman's Schoolgirls Problem.** Fifteen schoolgirls walk each day in five groups of three. Arrange the girls' walks for a week so that, in that time, each pair of girls walks together in a group just once. ■

$STS(9)$ gives a solution to the nine schoolgirls problem.

■ **Example 5.4 — Kirkman's Nine Schoolgirls Problem.** Nine schoolgirls walk for four days in three groups of three. Arrange the girls' walks for a week so that, in that time, each pair of girls walks together in a group just once.

The walking scheme is as follows:

Day 1: 123, 456, 789

Day 2: 147, 258, 369

Day 3: 159, 267, 348

Day 4: 357, 168, 249 ■

We can turn Steiner systems into an equivalent graph theory problem.

Definition 5.1.4 A *decomposition* of a graph G is a set of subgraphs that partition the edges of G.

Proposition 5.1.1 An $STS(n)$ is equivalent to a decomposition of K_n into triangles.

Theorem 5.1.2 Let $n > 0$. There exists an $STS(n)$ iff $n \equiv 1$ or 3 mod 6.

Proof of (⇒). Suppose there exists an $STS(n)$. We can consider when a decomposition of the edges of the complete graph K_n into triangles can exist.

Every vertex v must belong to $\frac{n-1}{2}$ triangles since $\deg(v) = n-1$.

Since the number of triangles is an integer, n must be odd, so $n \equiv 1,3,5 \mod 6$.

To rule out $n \equiv 5$ mod 6, for contradiction, assume $n = 6k+5$.

Since there are $\binom{n}{2}/3 = n(n-1)/6$ triangles in total, $n(n-1)/6$ must be a positive integer. Then

$$n(n-1)/6 = (6k+5)(3k+2)/3$$

is a positive integer. But neither $6k+5$ or $3k+2$ are divisible by 3. Contradiction.

$\therefore n \equiv 1,3$ mod 6. □

Part Two

Bibliography

Books

Index