Introduction to Topology

Notes from MATC27 Lecture

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Contents

Part One

1. Set Theory

1.1 Sets

1.1.1 Open and Closed sets

We start topology by considering open and closed sets learned in real analysis.

Definition 1.1.1 — Neighborhood. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A *neighborhood* of x (or x-nbhd) is a set $B_x = B(x, \varepsilon) = \{y \in \mathbb{R} \mid |x - y| < \varepsilon\} = (x - \varepsilon, x + \varepsilon)$.

Definition 1.1.2 — Open set. A set $S \subseteq \mathbb{R}$ is *open* in $\mathbb{R}$ iff $\forall x \in S, \exists B_x$ such that $B_x \subseteq S$.

This states that an open set has a neighborhood around any arbitrary point.
Alternatively, we can say a set is open iff $\forall x \in S, \exists \varepsilon_x > 0$ such that $x \in (x - \varepsilon_x, x + \varepsilon_x) \subseteq S$.

Definition 1.1.3 — Closed set. A set $F \subseteq \mathbb{R}$ is *closed* in $\mathbb{R}$ iff $\mathbb{R} \setminus F$ is open.

Again, we can alternatively state that F is closed iff $\forall y \notin F, \exists \varepsilon_y > 0$ such that $(y - \varepsilon_y, y + \varepsilon_y) \cap F = \varnothing$.

■ Example 1.1 — Examples of open sets.

(i) $\mathbb{R}$ is open.
(ii) $\varnothing$ is open in $\mathbb{R}$.
(iii) $\mathbb{R}$ and $\varnothing$ are both closed in $\mathbb{R}$.
(iv) Let $a, b \in \mathbb{R}, a < b$. The interval (a, b) is open in $\mathbb{R}$.
(v) $I = [0, 1]$ is closed in $\mathbb{R}$.

■

Proofs for Example 1.1.

(i) Let $x \in \mathbb{R}$. Choose $\varepsilon_x = 1 > 0$. Then $x \in (x - 1, x + 1) \subseteq \mathbb{R}$, since $x - 1, x + 1 \in \mathbb{R}$. $\therefore \mathbb{R}$ is open.
(ii) We know the empty set has no elements. By definition, S is open iff $\forall x \in S, \exists B_x$ such that $B_x \subseteq S$. When $S = \varnothing$, there is no such $x \in S$, so it is vacuously true. $\therefore \varnothing$ is open.

(iii) Consider $\mathbb{R} \setminus \mathbb{R} = \varnothing$. By (ii), $\varnothing$ is open. By Definition 1.1.3, $\mathbb{R}$ is closed.
Consider $\mathbb{R} \setminus \varnothing$. By (i), $\varnothing$ is open. Then $\varnothing$ is closed.

(iv) Let $x \in (a,b)$. Choose $\varepsilon_x = \min\{\frac{x-a}{2}, \frac{b-x}{2}\} > 0$. Then $x \in (x-\varepsilon_x, x+\varepsilon_x) \subseteq (a,b)$ by construction.

□

Note that both $\mathbb{R}$ and $\varnothing$ are both closed and open. These are called *clopen* sets. They are the only clopen sets in $\mathbb{R}$.

Exercise 1.1 Prove $\mathbb{R}$ and $\varnothing$ are the only clopen sets in $\mathbb{R}$. (Try using proof by contradiction.)

Exercise 1.2 Prove Example 1.1[v], i.e., that $I = [0,1]$ is closed.

■ **Example 1.2** $I = (0,1]$ is neither open nor closed. ■

Proof. First, we show I is not open.

We must show $\neg(\forall x \in I, \exists \varepsilon_x > 0, x \in (x-\varepsilon_x, x+\varepsilon_x) \subseteq I)$. So we should show $\exists x \in I, \forall \varepsilon_x > 0, x \in (x-\varepsilon_x, x+\varepsilon_x) \not\subseteq I$.

Choose $x = 1 \in I$.

Let $\varepsilon_x > 0$ be arbitrary.

Note $1+\varepsilon_x/2 > 1 \Rightarrow 1+\varepsilon_x/2 \notin (0,1] = I$. But $1+\varepsilon/2 \in (x-\varepsilon_x, x+\varepsilon_x)$ because $1+\varepsilon_x/2 < 1+\varepsilon_x$.

$\therefore (1-\varepsilon_x, 1+\varepsilon_x) \not\subseteq I$. □

Exercise 1.3 Show that I is also not closed.

However, consider that you can't just choose $\varepsilon_x/2$ because there are two cases: when $\varepsilon_x/2 \leq 1$ and when $\varepsilon_x/2 > 1$.

1.1.2 Building open and closed sets

Theorem 1.1.1 — Union and intersection of open and closed sets.
(i) The union of any (finite or infinite) collection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.
(ii) The intersection of a finite collection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.
(iii) The intersection of any (finite or infinite) collection of closed sets in $\mathbb{R}$ is closed in $\mathbb{R}$.
(iv) The union of a finite collection of closed sets in $\mathbb{R}$ is closed in $\mathbb{R}$.

■ **Example 1.3** The following examples show why 1.1.1(ii) and (iv) require the finite condition.

(i) Consider $I = [1/n, 1], n \in \mathbb{Z}^+$.
We can show I is closed in $\mathbb{R}$ by showing $\forall a,b \in \mathbb{R}, [a,b]$ is closed.
Then $\bigcup_{n=1}^{\infty} [1/n, 1] = (0,1]$ (whose equality we'd have to show with opposite set containment (and using the Archimedean property)).
In Example 1.2, we show this interval is not closed.
So 1.1.1(iv) requires the finite condition.

(ii) Consider $I = (0, 1+\frac{1}{n}), n \in \mathbb{Z}^+$. Then we showed in Example 1.1(iv) this interval is open.
Then $\bigcap_{n=1}^{\infty} (0, 1+1/n) = (0,1]$ (which we must show again).
So 1.1.1(ii) requires the finite condition.

■

R By Demorgan's Law, Theorem 1.1.1(i) and (iii) are equivalent, and Theorem 1.1.1(ii) and (iv) are equivalent. So it is sufficient to prove only two of these properties (such as (i) and (iv))

Proof of Theorem 1.1.1. (i) Let $\left\{A_j \mid A_j \subseteq \mathbb{R}, \text{open}\right\}, j \in J$, where J is the index set (which may be finite or infinite).
We want to show $A = \bigcup_{j \in J} A_j$ is open (by definition).
Let $x \in A$ be arbitrary.
Then $x \in A_{j_0}$ for some $j_0 \in J$ by definition of union.
So $\exists B_x$ such that $x \in B_x \subseteq A_{j_0}$ because A_{j_0} is open.
Since $A_{j_0} \subseteq A$ by definition of union, $B_x \subseteq A$.
$\therefore A$ is open.

(iv) Let $\{A_i \mid A_i \subseteq \mathbb{R}, \text{closed}\}, i \in \mathbb{Z}^+$.

□

Theorem 1.1.2 Every open set in $\mathbb{R}$ can be written as a countable (pairwise) disjoint union of open intervals. That is, for open $S \subseteq \mathbb{R}$, $a_j, b_j \in \mathbb{R}, a_j < b_j$

$$S = \bigsqcup_{j=1}^{\infty} (a_j, b_j)$$

where $(a_j, b_j) \cap (a_{j'}, b_{j'}) = \varnothing$ when $j \neq j'$.

Recall the following definition:

Definition 1.1.4 A point $x \in F$ is a *limit (or accumulation, or cluster) point* iff $\forall B_x, (B_x(x, \varepsilon) - \{x\}) \cap F \neq \varnothing$, i.e. any deleted neighborhood around x has points in F.

Theorem 1.1.3 F is closed in $\mathbb{R}$
$\Leftrightarrow$ Every convergent $\{x_n\} \subset F$ with limit l has $l \in F$
$\Leftrightarrow$ F contains all of its limit points.

Definition 1.1.5 — Open set in R^n. Let $S \subseteq \mathbb{R}^n$.
S is *open in R^n* iff $\forall x = (x_1, x_2, \ldots x_n) \in S, B(x, \varepsilon) = \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}, x \in B(x, \varepsilon) \subseteq S$.

Definition 1.1.6 — Closed in $\mathbb{R}^n$. Let $F \subseteq \mathbb{R}^n$.
F is *closed in $\mathbb{R}^n$* iff $\mathbb{R}^n - F$ is open in $\mathbb{R}^n$.

2. Topology

2.1 Introduction to topologies and topological spaces

2.1.1 Topology and topological space

Definition 2.1.1 — Topology. [Mun99, page 76] Let X be a nonempty set.

A *topology* $\mathcal{T}$ is a collection of subsets of X is a collection of subsets of X that satisfies:

(i) $\varnothing, X \in \mathcal{T}$

(ii) If $A_j \in \mathcal{T}, j \in \Lambda$ then $\bigcup_{j\in\Lambda} A_j \in \mathcal{T}$

(iii) If $A_j \in \mathcal{T}, j \in \Lambda, |\Lambda| < \infty$ then $\bigcap_{j\in\Lambda} A_j \in \mathcal{T}$

Consider how this looks like the union and intersection of open and closed sets in $\mathbb{R}$, except we use it as definition for a general set X.

Vocabulary 2.1 When $U \in \mathcal{T}$, we say U is *open* in X.

Definition 2.1.2 [Mun99, page 76] A *topological space* is $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a topology on X.

■ **Example 2.1** Consider $X = \mathbb{R}, \mathcal{T} = \{\text{open sets in } \mathbb{R}\}$. Then $(X, \mathcal{T})$ is a topological space. We call this the *standard topology* $\mathcal{T}_{\text{std}}$. ■

By Example 1.1(i) and (ii), the standard topology satisfies Definition 2.1.1(i). By Theorem 1.1.1, we have conditions (ii) and (iii).

Exercise 2.1 Show Example 2.1 satisfies Definition 2.1.1(iii).

■ **Example 2.2** Let $X = \mathbb{R}^n, \mathcal{T} = \{\text{open subsets of } \mathbb{R}^n\}$. Then $(X, \mathcal{T})$ is a topological space. ■

■ **Example 2.3** Let $X \neq \varnothing$. Let $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is a topological space. We call this the *trivial topology* (or indiscrete topology). ■

Proof. We are showing $\mathcal{T}$ is a topology.

(i) $\varnothing, X \in \mathcal{T}$ by definition of $\mathcal{T}$.

(ii) Let $A_j, j \in J$ be any collection of members of $\mathcal{T}$.
If $A_{j'} = X$ for some $j' \in J$, then $\bigcup_{j \in \Lambda} A_j \in \mathcal{T} = X \in \mathcal{T}$ by definition of $\mathcal{T}$.
If $A_j = \varnothing$ for all $j \in J$, then $\bigcup_{j \in \Lambda} A_j \in \mathcal{T} = \varnothing \in \mathcal{T}$ by definition of $\mathcal{T}$.

(iii) Let $A_j, j \in J, |J| < \infty$ be any collection of members of $\mathcal{T}$.
If $A_j = X$ for all $j \in J$, then $\bigcap_{j \in \Lambda} A_j \in \mathcal{T} = X \in \mathcal{T}$ by definition of $\mathcal{T}$.
If $A_{j'} = \varnothing$ for some $j' \in J$, then $\bigcap_{j \in \Lambda} A_j \in \mathcal{T} = \varnothing \in \mathcal{T}$ by definition of $\mathcal{T}$.

□

■ **Example 2.4** Let $X \neq \varnothing$. Let $\mathcal{T} = \mathcal{P}(X)$. Then $(X, \mathcal{T})$ is a topological space. We call this the *discrete topology*. ■

Exercise 2.2 Prove the discrete topology is a topology.

■ **Example 2.5** Let $X = \{a, b, c, d, e\}$.
Let $\mathcal{T}_1 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}\}$.
Let $\mathcal{T}_2 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.
Let $\mathcal{T}_3 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$.
Which of these $\mathcal{T}_i$ are topologies?
Consider $\{c, d\}, \{a, b, d, e\} \in \mathcal{T}_3$. However, $\{c, d\} \cap \{a, b, d, e\} = \{d\} \notin \mathcal{T}_3$. ■

Exercise 2.3 Show whether $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies or not.

We can depict topologies by drawing out the elements of X and the subsets of $\mathcal{T}$.

■ **Example 2.6** Let $X = \{a, b\}$. List all possible topologies on X.
We have the trivial topology $\mathcal{T}_1 = \{\varnothing, X\}$ and discrete topology $\mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b\}\} = \mathcal{P}(X)$.
There is also $\mathcal{T}_3 = \{\varnothing, X, \{a\}\}$ and $\mathcal{T}_4 = \{\varnothing, X, \{b\}\}$. ■

Exercise 2.4 Given $n \in \mathbb{N}$. Find the number of different topologies on a set X such that $|X| = n$.

2.1.2 Comparable Topologies

Definition 2.1.3 [Mun99, page 77] Let $X \neq \varnothing$. Suppose that $\mathcal{T}, \mathcal{T}'$ are two topologies on X.

(i) We say $\mathcal{T}'$ is *finer* than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$ (and strictly finer if the containment is strict).

(ii) We say $\mathcal{T}'$ is *coarser* than $\mathcal{T}$ if $\mathcal{T}' \subseteq \mathcal{T}$ (and strictly coarser if the containment is strict).

(iii) Two topologies are *comparable* if either $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

The terminology might be counter to intuition: "finer" meaning "smaller". However, consider a topological space as a pile of rocks. If we smash the rocks, there are now more rocks, though they are smaller, like how the topology is finer by having more elements.

■ **Example 2.7** Let $X = \{a, b\}$. Consider the trivial topology $\mathcal{T}_{\text{tri}} = \{\varnothing, X\}$ and the discrete topology $\mathcal{T}_{\text{dis}} = \{\varnothing, X, \{a\}, \{b\}\}$.

Then the discrete topology on X is strictly finer than the trivial topology on X.

We can see $\varnothing, X \in \mathcal{T}_{\text{dis}}$ and $\varnothing, X \in \mathcal{T}_{\text{tri}}$, so $\mathcal{T}_{\text{tri}} \subseteq \mathcal{T}_{\text{dis}}$. Since $\{a\} \in \mathcal{T}_{\text{dis}}$ and $\{a\} \notin \mathcal{T}_{\text{tri}}$. $\therefore \mathcal{T}_{\text{tri}} \subset \mathcal{T}_{\text{dis}}$. ■

All topologies are coarser than the trivial topology and finer than the discrete topology.

2.2 Basis for a topology

Definition 2.2.1 Let $(X, \mathcal{T})$ be a topological space.

A *basis*, β, for $\mathcal{T}$ is a collection of subsets of X such that:

(i) $\forall x \in X, \exists B \in \beta$ such that $x \in B$

(ii) If $\exists, B_1, B_2 \in \beta$ such that $x \in B_1 \cap B_2$, then $\exists B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Let $U \subset X$. We say that β *generates* $\mathcal{T}$ provided: if $\forall x \in U, \exists B_x \in \beta$ such that $x \in B_x \subseteq U$, then $U \in \mathcal{T}$ (i.e. U is open in $\mathcal{T}$).

This allows us to define topologies using a smaller collection of subsets instead of defining the topology directly. The definition for a basis generating a topology (and what is considered an open set) is parallel to the definition of open sets in $\mathbb{R}$ using ε-neighborhoods. Also, the definition requires an implication, but it is also a biconditional, i.e., if a set is open, then we can find a basis element for any arbitrary element in that set.

■ **Example 2.8** (a) Consider $(\mathbb{R}, \mathcal{T}_{\text{std}})$. Then $\beta = \{\,(a,b) \mid a,b \in \mathbb{R}\,\}$ is a basis.

(b) $\beta' = \{\,(a,b) \mid a,b \in \mathbb{Q}\,\}$ is also a basis for $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

(c) $\beta'' = \{\,(a,b) \mid a,b \in \mathbb{R} \setminus \mathbb{Q}\,\}$ is a basis too for $(\mathbb{R}, \mathcal{T}_{\text{std}})$.

(d) Let $X \neq \varnothing$. Consider $(X, \mathcal{T}_{\text{dis}})$. A basis for $\mathcal{T}_{\text{dis}}$ is $\beta = \{\,\{x\} \mid x \in X\,\}$.

■

Note that bases do not require the same cardinality; see (i) and (ii).

Proof for Example 2.8(a). We have to ensure the basis satisfies (i) and (ii) of Definition 2.2.1.

Let $x \in \mathbb{R}$. Let $\varepsilon = 1 > 0$. Then $x \in (x-1, x+1) \in \beta$, so condition (i) holds.

Suppose $\exists B_1(r_1) = (a,c), B_2(r_2) = (b,d)$ are open balls with radius r_1, r_2 such that $B_1 \cap B_2$. Then choose $r_3 = \min\{x-a, x-b, c-x, d-x\}$. So $x \in B_3(r_3) \subseteq B_1(r_1) \cap B_2(r_2)$ by choice of r_3. Then condition (ii) holds. (Alternatively, we can use Theorem 1.1.1 for this condition.) □

■ **Example 2.9** $\beta = \{\,(a,b) \mid a,b \in \mathbb{R}\,\}$ generates $\mathcal{T}_{\text{std}}$. ■

Proof. Recall $U \in \mathcal{T}_{\text{std}}$. We know from Theorem 1.1.1 that $U = \sqcup_{i=1}^{\infty}(a_i, b_i), a_i, b_i \in \mathbb{R}$.

Let $x \in U$ be arbitrary. So $x \in (a_i, b_i) \in \beta$ for some $i \in \mathbb{N}$.

$\therefore \forall x \in U, \exists B_x = (a_i, b_i)$ for some $i \in \mathbb{N}$ and $x \in B_x$. □

Now we want to check the $\mathcal{T}$ that a basis β generates is indeed a topology.

Proof. Suppose that:

(1) β is a basis for $\mathcal{T}$

(2) β generates $\mathcal{T}$

So now we have to prove Definition 2.1.1(i), (ii), and (iii).

(i) When $U = \varnothing \in X$, then $U \in \mathcal{T}$ because the definition of (2) vacuously holds.

So consider $U = X \in \mathcal{T}$ because of: Definition 2.2.1(i), i.e. (1); $U = X$; and β is a collection of subsets of X.

$\therefore \varnothing, X$ are open.

(ii) Let $A_j, j \in J$ be any collection of open sets on X.

Let $A = \cup_{j \in J} A_j \in \mathcal{T}$. Let $x \in A$ be arbitrary. Then $x \in A_j$ for some $j \in J$ by definition of union.
Since $A_j \in \mathcal{T}$ (i.e., A_j is open), so $\exists B_x \in \beta$ such that $x \in B_x \subseteq A_j$ by (2).
Thus $x \in B_x \subseteq A$ because $A_j \subseteq \cup_{j \in J} A_j \in \mathcal{T} = A$.
$\therefore A$ is open.

(iii) Let $A_1, A_2, \ldots, A_n \in \mathcal{T}$.
BASE CASE: $A_1, A_2 \in \mathcal{T}$. Let $x \in A_1 \cap A_2$ be arbitrary.
We have that $x \in A_1$ and $x \in A_2$ by definition of intersection. Then $\exists B_1, B_2 \in \beta$ such that $x \in B_1 \subseteq A_1$ and $x \in B_2 \subseteq A_2$ by $A_1, A_2 \in \mathcal{T}$.
We use Definition 2.2.1(ii) i.e. (1) to give that $\exists B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$.
INDUCTION STEP: Recall $A_1, \ldots, A_n \in \mathcal{T}$. Assume $A_1 \cap \ldots \cap A_{n-1} \in \mathcal{T}$.
Consider $A_1 \cap \ldots \cap A_{n-1} \cap A_n = (A_1 \cap \ldots \cap A_{n-1}) \cap A_n \in Tau$ since we have already proved the intersection of two open sets are open.

□

The definition for a basis generating a topology is not an intuitive or constructive way of creating a topology. The following lemma shows us how we get a topology from a basis.

2.2.1 Topology generation from basis

Theorem 2.2.1 — Topology from Basis Lemma. [Mun99, page 80] Let $X \neq \varnothing$. Let β be a basis for X (i.e., that generates some τ on X). Then $\mathcal{T} = \{ U \in X \mid U = \cup_{x \in U} B_x$, for some $B_x \in \beta \}$.

To prove this, we show opposite set containment.

Exercise 2.5 Prove Lemma 2.2.1.

Note that $U = \cup_{x \in U} B_x$, for some $B_x \in \beta$ is not unique.

■ **Example 2.10** Consider the topological space $(\mathbb{R}, \mathcal{T}_{\text{std}})$ with basis $\beta = \{ (a,b) \mid a,b \in \mathbb{R}, a < b \}$.
We have that $U = (0,1) \cup (\pi, 7) \in \mathcal{T}_{\text{std}}$. Notice $(0,1) \in \beta$ and $(\pi, 7) \in \beta$.
However, we also have $U = (0,1) \cup (\pi, 4) \cup (3.5, 7)$. ■

Definition 2.2.2 — Topologizing. The action of creating a topological space (X, τ_β) from (X, β) is called *topologizing* X.

However, we want the converse: to be able to find bases given a topological space.

Theorem 2.2.2 — Basis Check Lemma. [Mun99, page 80] Let X be a topological space. Suppose $\mathcal{C}$ is a collection of open sets in X such that $\forall U \in \mathcal{T}_X$ and $\forall x \in U, \exists C \in \mathcal{C}$ such that $x \in C \subseteq U$.
Thus $\mathcal{C}$ is a basis and $\mathcal{C}$ generates $\mathcal{T}_X$.

To prove this, we need to show two things: that $\mathcal{C}$ is a basis and that $\mathcal{C}$ generatesz $\mathcal{T}_X$.
The cover condition ($\forall x \in X, \exists C \in \mathcal{C}, x \in C$) in the definition of basis follows if we just take $U = X$.
Then we have to show the refinement condition in the definition of basis (if $C_1, C_2 \in \mathcal{C}$ such that $x \in C_1 \cap C_2$, then $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$).
In this proof, we will have a method of actually constructing the topology.

■ **Example 2.11** Let $X = \{a,b,c\}$. Let $A = \{\{a\},\{b\}\} \subset \mathcal{P}(X)$.
Find a basis for some topology $\mathcal{T}_A$ such that $A \subseteq \mathcal{T}_A$ (i.e., the members of A are open sets in

X).

Consider $\tau_A = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$.

The plan is to find all find all finite intersections of members in A.

- $\{a\} \cap \{a\} = \{a\}$. $\therefore \{a\} \in \mathcal{C}$.
- $\{b\} \cap \{b\} = \{b\}$. $\therefore \{b\} \in \mathcal{C}$.
- $\{a\} \cap \{b\} = \varnothing$. $\therefore \varnothing \in \mathcal{C}$.
- $\cap_{\alpha \in \varnothing} A_\alpha = X = \{a,b,c\}$. $A_\alpha \in A$. $\therefore \in \mathcal{C}$.
  Recall $\{A_\alpha\} \subset \mathcal{P}(X)$. Then $\cup_{\alpha \in J} A_\alpha = \{x \in X \mid x \in A_\alpha$ for some $\alpha \in J\}$.
  Also $\cap_{\alpha \in J} A_\alpha = \{x \in X \mid x \in A_\alpha$ for each $\alpha \in J\}$. When we intersect more things, we get a smaller set. So the reverse is true (by the precedent being vacuously true): the empty intersection is the full set.

So this $\mathcal{C} = \{\varnothing, X, \{a\}, \{b\}\}$ is a basis. ■

Exercise 2.6 Use Lemma 2.2.2 to show $\mathcal{C}$ is a basis.

Definition 2.2.3 — Equivalent basis. Let $X \neq \varnothing$. Let β and β' are basis on X.

We say that β and β' are *equivalent* if $\mathcal{T}_\beta = \mathcal{T}_{\beta'}$.

Theorem 2.2.3 — Basis equivalence. Let $X \neq \varnothing$. Let β and β' be basis on X. Then β' and β are equivalent iff

(a) $\forall B \in \beta$ and $\forall x \in B$, $\exists B' \in \beta'$ such that $x \in B' \subseteq B$

(b) $\forall B' \in \beta'$ and $\forall x \in B'$, $\exists B \in \beta$ such that $x \in B \subseteq B'$

We say two basis are equivalent if every basis element in β can be built from basis elements in β', and vice versa. When we say "built", we mean that all points covered by a basis element can be covered by basis elements from the other basis.

R If only Theorem 2.2.3(a) holds, then $\mathcal{T}_\beta \subseteq \mathcal{T}_{\beta'}$, i.e., $\mathcal{T}_{\beta'}$ is finer (larger, more) than $\mathcal{T}_\beta$. If only Theorem 2.2.3(b) holds, then $\mathcal{T}_{\beta'} \subseteq \mathcal{T}_\beta$, i.e., $\mathcal{T}_{\beta'}$ is coarser than $\mathcal{T}_\beta$.

2.2.2 Subbasis

Definition 2.2.4 — Subbasis. [Mun99, page 82] Let $(X, \mathcal{T})$ be a topological space.

A *subbasis* $\mathcal{S}$ is a collection of open sets of X such that

$$\forall U \in \mathcal{T}, \quad U = \bigcup_{\alpha \in J} \left(\bigcap_{i=1}^{n_\alpha} S_i \right)$$

where $S_i \in \mathcal{S}$.

That is, for any open set $U = \{$unions of all finite intersections of elements in $\mathcal{S}\}$. So Example 2.11 is an example of subbasis.

Then $B_1 = S_1 \cap S_2 \cap \cdots \cap S_n$ for fixed n is a basis element.

■ **Example 2.12** (i) Consider $(\mathbb{R}, \mathcal{T}_{\text{std}})$. Then $\beta = \{(a,b) \mid a,b \in \mathbb{R}, a < b\}$.
Note $(a,b) = (-\infty, b) \cap (a, \infty)$.
So then we get $\mathcal{S} = \{$all unbounded open intervals in $\mathbb{R}\}$.

(ii) Consider $(\mathbb{R}^2, \mathcal{T}_{\text{std}})$. Then $\beta = \{$all open rectangles with parallel sides$\}$. What is a subbasis $\mathcal{S}$? We have $\mathcal{S} = \{(x,y) \mid a < x < b,\ a,b \in \mathbb{R}\} \cup \{(x,y) \mid c < y < d,\ c,d \in \mathbb{R}\}$.

■

2.3 Subspace Topology

Definition 2.3.1 Let $(X, \mathcal{T}_X)$ be a topological space. Let $Y \subseteq X, Y \neq \varnothing$.
A set $V \subset Y$ is *open in* Y iff $V = Y \cap U$ for some $U \in \mathcal{T}_X$.

■ **Example 2.13** Consider $(\mathbb{R}, \mathcal{T}_{\text{std}})$. Let $Y = (0,1] \subseteq \mathbb{R}$ (but note Y is not open). Prove $(\frac{1}{2}, 1]$ is open in Y. ■

Proof. We have that $V \subseteq Y, V \neq \varnothing$.
Observe $V = (\frac{1}{2}, 1] = (0,1] \cap (\frac{1}{2}, 2)$, so it's open in Y but not open in $\mathcal{T}_{\text{std}}$. □

R An open set in a subset Y need not be open in X.

Definition 2.3.2 — Subspace Topology. [Mun99, page 88] Let $(X, \mathcal{T}_X)$ be a topological space. Let $Y \subseteq X, Y \neq \varnothing$.
The *subspace topology* (or relative, or induced) on Y is

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}$$

The pair $(Y, \mathcal{T}_Y)$ is called a *subspace*.

Note that Munkres doesn't show that this is a topology, so this should be a theorem. We will prove it is a topology.

Proposition 2.3.1 $\mathcal{T}_Y$ is a topology on Y.

Proof. (i) $\varnothing = Y \bigcap \varnothing$ (and $\varnothing \in T_X$ topology). $\therefore \varnothing \in \mathcal{T}_Y$.
$Y = Y \bigcap X$ (and $X \in T_X$ topology). $\therefore Y \in \mathcal{T}_Y$

(ii) Suppose $Y \bigcap U_\alpha, \forall \alpha \in J$ are open in Y, i.e. $Y \bigcap U_\alpha \in \mathcal{T}_Y$.

$$\bigcup_{\alpha \in J} (Y \bigcap U_\alpha) = Y \bigcap \underbrace{\Big(\bigcup_{\alpha \in J} U_\alpha \Big)}_{\mathcal{T}_X} \in \mathcal{T}_Y$$

We know that $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_X$ because of arbitrary union in a topological space.
$\therefore \bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}_X$.

(iii) We prove the finite intersection is an open set by induction.

BASE CASE: Let $Y \cap U_1, Y \cap U_2 \in \mathcal{T}_Y$.
Consider $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2)$ (by commutativity, associativity of intersection).
We have that $U_1 \cap U_2 \in T_X$ (because $\mathcal{T}_X$ is a topology and finite intersection of open sets is open). $\therefore Y \cap (U_1 \cap U_2) \in \mathcal{T}_Y$.
INDUCTION STEP: Let $Y \cap U_1, \ldots, Y \cap U_n \in \mathcal{T}_Y$. Assume $\bigcap_{i=1}^{n-1} (Y \cap U_i) \in \mathcal{T}_Y$.
Then consider $\bigcap_{i=1}^{n} (Y \cap U_i) = \bigcap_{i=1}^{n-1} (Y \cap U_i) \cap (Y \cap U_n)$, which are both open sets in Y. By the base case, $\bigcap_{i=1}^{n-1} (Y \cap U_i) \cap (Y \cap U_n) \in \mathcal{T}_Y$

$\therefore (Y, \mathcal{T}_Y)$ is a topological space. □

■ Example 2.14 Let $X = \{a,b,c,d,e,f\}$ with $\mathcal{T}_X = \{X, \varnothing, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e,f\}\}$.

Let $Y = \{a,c,e\}$. What is $\mathcal{T}_Y$?

Recall $\mathcal{T}_Y = \{\, Y \cap U \mid U \in \mathcal{T}_X \,\}$.

$\mathcal{T}_Y = \{Y, \varnothing, \{a\}, \{c\}, \{a,c\}, \{c,e\}\}$. Note that $\{a,c\} = Y \cap \{a,c\} = Y \cap (\{a\} \cup \{c\})$ (even though we are already given $\{a,c\}$, some topologies may only be defined in by their basis, so we might need to take the union and intersect Y with that). ■

■ Example 2.15

(i) Let $(\mathbb{R}_2, \mathcal{T}_{\text{std}})$.
Let $Y = \left\{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 9 \,\right\} \subseteq \mathbb{R}^2$ with the subspace topology is a subspace.

(ii) Consider $Y = \left\{\, (x,y,z) \mid x^2 + y^2 + z^2 = 1 \,\right\}$. It is a subspace with the subspace topology.

■

2.3.1 Subspace basis

Theorem 2.3.2 Let $(X, \mathcal{T}_X)$ be a topological space. Let $(Y, \mathcal{T}_Y)$ be a subspace. If β_X is a basis for $\mathcal{T}_X$, then a basis for $\mathcal{T}_Y$ is $\beta_Y = \{\, Y \cap B \mid B \in \beta_X \,\}$.

Proof. We use Lemma 2.2.2.

We want to show $\forall Y \cap U \in \mathcal{T}_Y$ and $\forall y \in Y \cap U$, $\exists C \in \beta_Y$ such that $y \in C \subseteq Y \cap U$.

Let $Y \cap U \in \mathcal{T}_Y$ be an arbitrary open set (so U is any open set in X, i.e., $U \in \mathcal{T}_X$).

Let $y \in Y \cap U$ be arbitrary. So by definition of intersection, $y \in Y$ and $y \in U$. Because U is open, and β_X generates $\mathcal{T}_X$, $\exists B \in B_X$ such that $y \in B \subseteq U$.

Then $B \cap Y \subseteq U \cap Y$. Choose $C = B \cap Y \in \beta_Y$. □

2.4 Finite Product Spaces

Section 15 of [Mun99] covers the $n = 2$ case, and Section 17 covers the general case.

Recall a finite collection of nonempty sets X_i, $i = 1, \ldots, n$. Then

$$X_1 \times X_2 \times \cdots \times X_n = \{\, (x_1, x_2, \ldots, x_n) \mid x \in X_i, i = 1, \ldots, n \,\}$$

Definition 2.4.1 — Finite product topology. [Mun99, page 86] Let $(X_1, \mathcal{T}_1), \ldots (X_n, \mathcal{T}_n)$ be a finite collection of topological spaces.

The *product topology*, $\mathcal{T}_{\text{prod}}$ on $X_1 \times \cdots \times X_n$ is the topology generated by the basis

$$\beta_{\text{prod}} = \{\, U_1 \times U_2 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i \,\}$$

The pair $(X_1 \times \cdots \times X_n, \mathcal{T}_{\text{prod}})$ is the *product space*.

■ Example 2.16 Consider $X = [a,b]$ is a subspace of $\mathbb{R}$, and $Y = \left\{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 9 \,\right\}$ is a subspace of $\mathbb{R}^2$.

Then $X \times Y$ is a cylinder with radius 3 and height $b - a$. ■

Proposition 2.4.1 $\beta_{\text{prod}} = \{\, U_1 \times U_2 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i \,\}$ is a basis.

Proof. Note $X_i \in \mathcal{T}_i$ for each $i = 1, \ldots, n$.

Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$.

Choose $B = (X_1 \times X_2 \times \cdots \times X_n) \in \beta_{\text{prod}}$ by the previous note.

$\therefore \boldsymbol{x} \in B$.

Suppose $B_1, B_2 \in \beta_{\text{prod}}$ and $\boldsymbol{x} \in B_1 \cap B_2$.

Then $B_1 \cap B_2 = (U_1 \times \cdots \times U_n) \cap (O_1 \times \cdots \times O_n)$, where $U_i \in \mathcal{T}_i$, $O_i \in \mathcal{T}_i$ by definition of $B_i \in \beta_{\text{prod}}$.

$$\begin{aligned} B_1 \cap B_2 &= (U_1 \times \cdots \times U_n) \cap (O_1 \times \cdots \times O_n) \\ &= (U_1 \cap O_1) \times \cdots \times (U_n \cap O_n) \\ &\in \beta_{\text{prod}} \end{aligned}$$

Each $(U_i \cap O_i) \in \mathcal{T}_i$ by closure under finite intersection.

So choose $B_3 = B_1 \cap B_2$.

$\therefore x \in B_3 \subseteq B_1 \cap B_2$.

□

Theorem 2.4.2 [Mun99, page 86] Let $\beta_1, \beta_2, \ldots, \beta_n$ be basis for $X_1, \ldots, X_n$ (i.e., β_i generates X_i). Then $\beta = \{B_1 \times B_2 \times \cdots \times B_n \mid B_i \in \beta_i\}$ is a basis for $\mathcal{T}_{\text{prod}}$ on $X_1 \times \cdots \times X_n$.

There are multiple ways to prove this. We can prove this by definition—show it's a basis and show the basis generates $\mathcal{T}_{\text{prod}}$; we can show β_{prod} and this proposition's β are equivalent by Theorem 2.2.3; or we can apply Lemma 2.2.2. The latter is the shortest proof.

2.4.1 Projection

Definition 2.4.2 — Projection map. [Mun99, pages 87, 114] Let $X_1, \ldots, X_n, \ldots$ (so it can be finite or infinite) be a collection of topological spaces, i.e. $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in J}$.

Then the map

$$\pi_i : X_1 \times \cdots \times X_i \times \cdots \times X_n \times \cdots \to X_i \qquad \text{(Countable case)}$$
$$\pi_\gamma : \prod_{\alpha \in J} X_\alpha \to X_\gamma \qquad \text{(General case)}$$

defined by

$$\pi_i\big((x_1, \ldots, x_i, \ldots, x_n, \ldots)\big) = x_i$$
$$\pi_\gamma\big((x_\alpha)_{\alpha \in J}\big) = x_\gamma$$

is called the *projection map* onto the i-th component or the component associated to the $\gamma \in J$ index.

■ **Example 2.17** (a) Consider the projection map $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $(x, y) \mapsto x$. This flattens the plane into the x-axis.

(b) Consider $\pi_1 : (0,1) \times \mathbb{R} \to (0,1)$ defined by $(x, y) \mapsto x$ where $x \in (0,1), y \in \mathbb{R}$. We have an open rectangular vertical strip, which just becomes a point between 0 and 1.

■

Theorem 2.4.3 [Mun99, page 88] Let $X_1, \ldots, X_n$ be a finite collection of topological spaces.

The set

$$\mathcal{S} = \{S_1, S_2, \dots\} = \bigcup_{i=1}^{n} \{S_i\}$$

where $\{S_i\} = \mathcal{S}_i = \left\{ \pi_i^{-1}(U_i) \mid U_i \in \mathcal{T}_i \right\}$, is a subbasis for $\mathcal{T}_{\text{prod}}$ on $X_1 \times \cdots \times X_n$.

R This union $\bigcup_{i=1}^{n} S_i$ is only for notation, that is, these S_i could be of different types, but we are not taking the literal set notation definition, but taking it as a collection.

This $\mathcal{S}$ is a collection of subsets (as a subbasis is). Each subset is made of the preimage of the projection of each open set in each of the original topologies.

$\mathcal{S}$ is not a basis in general, but the basis $\beta_{\mathcal{S}} = \bigcap_{\text{finite}} S_i$ are created from all possible finite intersections of elements in $\mathcal{S}$.

■ **Example 2.18** Let $X = \{a, b, c\}$ with $\mathcal{T}_X = \{X, \varnothing, \{a\}, \{a, c\}\}$.
Let $Y = \{u, v\}$ with $\mathcal{T}_Y = \{Y, \varnothing, \{u\}\}$.

(a) We want to find the subbasis $\mathcal{S}$ for $\mathcal{T}_{\text{prod}}$ on $X \times Y$.
Then $X \times Y = \{(a,u), (a,v), (b,u), (b,v), (c,u), (c,v)\}$.
There are two projection maps, $\pi_X : X \times Y \to X$ defined by $(x,y) \mapsto x$ and $\pi_Y : X \times Y \to Y$ defined by $(x,y) \mapsto y$.
So $\mathcal{S} = \left\{ \pi_X^{-1}(U_i), \pi_Y^{-1}(V_i) \mid U_i \in \mathcal{T}_X, V_i \in \mathcal{T}_Y \right\}$.
Now we compute the preimages (which becomes each $S_i \in \mathcal{S}$).

$$\begin{aligned}
S_1 = \pi_X^{-1}(X) &= \{ (x,y) \in X \times Y \mid \pi_X((x,y)) \in X \} && \text{(Def of preimage)} \\
&= X \times Y = \pi_Y^{-1}(Y) \\
S_2 = \pi_X^{-1}(\varnothing) &= \varnothing = \pi_Y^{-1}(\varnothing) \\
S_3 = \pi_X^{-1}(\{a\}) &= \{(a,u), (a,v)\} \\
S_4 = \pi_X^{-1}(\{a,c\}) &= \{(a,u), (a,v), (c,u), (c,v)\} \\
S_5 = \pi_Y^{-1}(\{u\}) &= \{(a,u), (b,u), (c,u)\}
\end{aligned}$$

Then we get the subbasis $\mathcal{S}$,

$$\begin{aligned}
\mathcal{S} &= \left\{ \pi_X^{-1}(U_i), \pi_Y^{-1}(V_i) \mid U_i \in \mathcal{T}_X, V_i \in \mathcal{T}_Y \right\} \\
&= \{S_1, S_2, S_3, S_4, S_5\} \\
&= \{X \times Y, \varnothing, \{(a,u), (a,v)\}, \\
&\quad \{(a,u), (a,v), (c,u), (c,v)\}, \\
&\quad \{(a,u), (b,u), (c,u)\}\}
\end{aligned}$$

(b) We want to find the basis $\beta_{\mathcal{S}}$ generated by $\mathcal{S}$.
We take all possible finite intersections (by taking all pairwise intersections, which considers any other finite intersections).

$$\begin{aligned}\beta_S = \{X \times Y, \varnothing, \{(a,u),(a,v)\} &= S_3,\\ \{(a,u),(a,v),(c,u),(c,v)\} &= S_4,\\ \{(a,u),(b,u),(c,u)\} &= S_5\\ \{(a,u)\} &= S_3 \cap S_5\\ \{(a,u),(c,u)\} &= S_4 \cap S_5\}\end{aligned}$$

■

2.5 Infinite Product Spaces

Recall that for a finite collection of topological spaces $(X_1, \mathcal{T}_1), \dots (X_n, \mathcal{T}_n)$, the product topology $\mathcal{T}_{\text{prod}}$ on $X_1 \times \cdots \times X_n$ is the topology generated by the basis $\beta_{\text{prod}} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i\}$.

Also recall Theorem 2.4.3, which states that $\beta = \{B_1 \times B_2 \times \cdots \times B_n \mid B_i \in \beta_i\}$ is an equivalent basis.

One more equivalent construction of the product space is in Theorem 2.4.2, giving the subbasis $\mathcal{S} = \bigcup_{i=1}^{n} \{S_i\}$ where $\{S_i\} = \mathcal{S}_i = \left\{\pi_i^{-1}(U) \mid U \in \mathcal{T}_{X_i}\right\}$. (That is, the subbasis elements are all the preimages of projections of open sets of each factor topology.)

This final construction will be important for infinite product spaces.

2.5.1 Box topology

Consider an infinite collection of topological spaces $X_1 \times X_2 \times \cdots$. Intuitively, each element in this collection is an infinite tuple.

Definition 2.5.1 — Box Topology. Let J be an index set. Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces. The *box topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the basis

$$\beta = \left\{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \in \mathcal{T}_{X_\alpha}\right\}.$$

See that it's called the "box" topology because we get these boxes of open sets. For this to be rigorously defined, we need to define what it means to index and take the product over an infinite set.

Definition 2.5.2 — J-tuple. [Mun99, page 113] Let J be an index set. Let $X \neq \varnothing$. A *J-tuple* of elements of X is a function $x : J \to X, \alpha \mapsto x(\alpha)$. (We notate this as x_α, the α coordinate of x.)

So we get the tuple $\left((x_\alpha)\right)_{\alpha \in J}$. Now we can precisely define the Cartesian product for infinite collections.

Definition 2.5.3 — Cartesian product. Let $\{X_\alpha\}_{\alpha \in J}$. The *Cartesian product* $\prod_{\alpha \in J} X_\alpha$ is the set of all J-tuples $\left((x_\alpha)\right)_{\alpha \in J}$ of elements in $\bigcup_{\alpha \in J} X_\alpha \ (= X)$ such that $x_\alpha \in X_\alpha$ for each $\alpha \in J$.

Theorem 2.5.1 [Mun99, page 116] The box topology is generated by the basis

$$\beta = \left\{\prod_{\alpha \in J} B_\alpha \mid B_\alpha \in \beta_{X_\alpha}\right\}.$$

Once again, we can use Lemma 2.2.2 to show this basis generates the box topology.

Now, we consider an alternate topology for this infinite product of topological spaces, using Theorem 2.4.2

2.5.2 Infinite product topology

Definition 2.5.4 — Product topology. [Mun99, page 114] Let J be an index set. Let $\{X_\alpha\}_{\alpha\in J}$ be a collection of topological spaces.

The *product topology* $\mathcal{T}_{\text{prod}}$ on $\prod_{\alpha\in J} X_\alpha$ is the topology generated by the subbasis

$$\mathcal{S} = \bigcup_{\alpha\in J} \mathcal{S}_\alpha$$

where $\mathcal{S}_\alpha = \{\, \pi_\alpha^{-1}(U) \mid U \in \mathcal{T}_{X_\alpha} \,\}$.

R If the index set J is finite, the definition for the box and product topology are equivalent. But if J is infinite, the box topology is finer (i.e., larger) than the product topology.

We prefer the product topology $\mathcal{T}_{\text{prod}}$ because many theorems only apply to this more restricted topology.

■ **Example 2.19** Let $(X,\mathcal{T}_i)$ for $i = 1,2,3$ be topological spaces, where each $\mathcal{T}_i$ is the discrete topology. Prove $X_1 \times X_2 \times X_3$ has the discrete topology. ■

Proof. We know that $\mathcal{T}_i$ is generated by basis $\beta_i = \{\, \{x_i\} \mid x_i \in X_i \,\}$.

Also, $\mathcal{T}_{\text{prod}}$ is generated by basis β_{prod} such that

$$\begin{aligned}\beta_{\text{prod}} &= \{\, B_1 \times B_2 \times B_3 \mid B_i \in \beta_i,\ i = 1,2,3 \,\}\\ &= \{\, \{x_1\} \times \{x_2\} \times \{x_3\} \mid x_i \in X_i \,\}\\ &= \{\, \{(x_1,x_2,x_3)\} \mid x_i \in X_i \,\}\end{aligned}$$

$\therefore$ $\mathcal{T}_{\text{prod}}$ is the discrete topology on $X_1 \times X_2 \times X_3$. □

2.6 Continuous functions

Note that when we consider a function $f : X \to Y$ from one topological space to another, it induces a map $f' : \mathcal{P}(X) \to \mathcal{P}(Y)$.

Definition 2.6.1 — Continuous function. [Mun99, page 102] Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a function.

We say that f is *continuous* if $\forall V \in \mathcal{P}(Y)$, if V is open in Y, then $f^{-1}(V)$ is open in X. (That is, $V \in T_y \Rightarrow f^{-1}(V) \in \mathcal{T}_X$.)

Recall that the preimage $f^{-1}(V) = \{\, x \in X \mid f(x) \in V \,\} \subseteq X$.

Proposition 2.6.1 — Constant functions are continuous. Let X,Y be topological spaces.

Let $f : X \to Y$ be a constant function, i.e., For some $a \in Y$, $\forall x \in X, x \mapsto a$.

Then f is continuous.

Proof. Let $V \in \mathcal{T}_Y$ be arbitrary.
We consider two cases:

$$f^{-1}(V) = \begin{cases} X, & \text{if } a \in V \\ \varnothing, & \text{if } a \notin V \end{cases}$$

By trivial open sets in a topology, $\varnothing, X \in \mathcal{T}_X$ because X is a topological space.
$\therefore f$ is continuous. □

■ **Example 2.20** Let $X = \{a,b,c,d\}$ and $Y = \{x,y,z,w\}$.
Suppose $\mathcal{T}_X = \{X, \varnothing, \{a\}, \{a,b\}, \{a,b,d\}\}$ and $\mathcal{T}_Y = \{Y, \varnothing, \{x\}, \{y\}, \{x,y\}, \{x,y,z\}\}$.
Prove or disprove $f : X \to Y$ is continuous, where

$$\begin{aligned} a &\mapsto x \\ b &\mapsto x \\ c &\mapsto z \\ d &\mapsto w \end{aligned}$$

■

Proof. We claim that f is not continuous.
Consider $\{x,y,z\} \in \mathcal{T}_Y$.
We compute $f^{-1}(\{x,y,z\}) = \{a,b,c\} \notin \mathcal{T}_Y$ by definition of f. □

Exercise 2.7 With the same topological spaces X, Y in Example 2.20, consider the map

$$\begin{aligned} a &\mapsto y \\ b &\mapsto z \\ c &\mapsto w \\ d &\mapsto z \end{aligned}$$

Proposition 2.6.2 [Mun99, page 107] The composition of continuous maps are continuous.

Proof. Let X, Y, Z be topological spaces. Let $g : X \to Y, f : Y \to Z$.
Suppose f, g are continuous. We want to show $f \circ g$ is continuous.
We have that $X \xrightarrow{g} Y \xrightarrow{f} Z$ and $X \xrightarrow{f \circ g} Z$.
Consider that $(f \circ g)^{-1}_{(X)} = (g^{-1} \circ f^{-1})_{(X)}$ by shoes-socks.
Also, since f, g are continuous, we have that for any $V \in \mathcal{T}_Z$, $f^{-1}(V) \in \mathcal{T}_Y$ and for any $U \in \mathcal{T}_Y$, $g^{-1}(U) \in \mathcal{T}_X$.
Let $V \in \mathcal{T}_Z$.

$$\begin{aligned} (f^{-1} \circ g^{-1})(V) &= g^{-1}(f^{-1}(V)) \\ &= g^{-1}(U),\ U \in \mathcal{T}_Y \\ &= W \in \mathcal{T}_X \end{aligned}$$

$\therefore$ The composition of continuous functions are continuous. □

Definition 2.6.2 — Open map. Let X,Y be topological spaces. Let $f : X \to Y$ be a map.
We say that f is open if f maps open sets in X to open sets in Y.

Note this definition can be considered as a sort of converse to continuous maps in topological spaces, and imporantly, they are not equivalent.

■ **Example 2.21** Consider $X = Y = \mathbb{R}$ with the standard topology.

From Proposition 2.6.1, we know that for some constant $a \in \mathbb{R} = Y$, and $\forall x \in \mathbb{R} = X$, that $f(x) = a$ is continuous.

Consider $f(\mathbb{R}) = \{f(x) \mid x \in \mathbb{R}\} = \{a\}$.

Then we could show that $\{a\}$, i.e., any singleton set in $(\mathbb{R}, \mathcal{T}_{\text{std}})$ is closed. To show this, we show that $\mathbb{R} - \{a\} = (-\infty, a) \cup (a, \infty)$.

We have that

$$(-\infty,a)\cup(a,\infty) = \Big(\bigcup_{n\in\mathbb{N}} \underbrace{(-n+a,a)}_{\in\beta_{\text{std}}}\Big) \cup \Big(\bigcup_{n\in\mathbb{N}} \underbrace{(a,n+a)}_{\in\beta_{\text{std}}}\Big)$$

which is the union of open sets (specifically, basis elements).

$\therefore \{a\}$ is closed.

The only clopen sets in $(\mathbb{R}, \mathcal{T}_{\text{std}})$ are $\varnothing, \mathbb{R}$.

$\therefore \{a\}$ is not open, so the map is not open. ■

Theorem 2.6.3 [Mun99, page 104] Let X,Y be a topological spaces.
Let $f : X \to Y$ be a map. Then f is continuous iff for each $B \in \beta_Y, f^{-1}(B) \in \mathcal{T}_X$.

Proposition 2.6.4 — Projection maps are continuous. Let X,Y be topological spaces. Then the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ defined by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ are continuous.

Proof. Let $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ be open sets.

We compute the preimage $\pi_1^{-1}(U) = U \times Y$. We have that $U \in \mathcal{T}_X$ (as given) and $Y \in \mathcal{T}_Y$ (Y is a topology). Then $U \times Y \in \beta_{\text{prod}}$ (by Definition 2.4.1).

Since $\beta_{\text{prod}} \subseteq \mathcal{T}_{\text{prod}}$, then $U \times Y$ is open.

$\therefore \pi_1$ is continuous.

Now we have that $\pi_1^{-1}(V) = \underbrace{X}_{\in\mathcal{T}_X} \times \underbrace{V}_{\in\mathcal{T}_Y} \in \beta_{\text{prod}} \subseteq \mathcal{T}_{\text{prod}}$.

$\therefore \pi_2$ is continuous. □

This theorem not only holds in the general finite case, but also in the infinite case.

Exercise 2.8 Prove the projection maps on $\pi_{\alpha\in J} X_\alpha$.

Proposition 2.6.5 [Mun99, page 110] Let X,Y,Z be topological spaces.
Let $f : Z \to X \times Y$ defined by $f(z) = (f_1(z), f_2(z))$ where $f_1 : Z \to X$ and $f_2 : Z \to Y$.
Then f is continuous iff f_1 and f_2 are continuous.

Proof. Suppose f is continuous.

Let $U \in Tau_X$ and $V \in \mathcal{T}_Y$ be open sets. Notice that $f_1 = \pi_1 \circ f$. We know the projection map is continuous (Prop 2.6.4) and the composition of continuous maps is continuous (Prop 2.6.2). Similarly, $f_2 = \pi_2 \circ f$.

$\therefore f_1, f_2$ are continuous.

Suppose f_1, f_2 are continuous.

Let $U \times V \in \mathcal{T}_{\text{prod}}$. Then $U \in Tau_X$ and $V \in \mathcal{T}_Y$.

Compute $f^{-1}(U \times V)$. We want $a \in f^{-1}(U \times V)$. So $f(a) \in U \times V$. Then $f_1(a) \in U$ and $f_2(a) \in V$.

So $f_1^{-1}(U \times V) = \underbrace{f_1^{-1}(U)}_{\in \mathcal{T}_Z} \cap \underbrace{f_2^{-1}(V)}_{\in \mathcal{T}_Z} \in \mathcal{T}_Z$ by closure under finite intersection. □

2.7 Homeomorphism

Definition 2.7.1 [Mun99, page 105] Let X, Y be topological spaces. A function $f : X \to Y$ is a homeomorphism if f is bijective and continuous, and its inverse f^{-1} is continuous.

We say X and Y are *homeomorphic*, $X \cong Y$, if there exists a homeomorphism between them.

A homeomorphism shows when two topological spaces have an equivalent structure, i.e., the homeomorphism preserves the topology. For example, if two spaces are topologically homeomorphic, you can continuously deform one to the other.

Proposition 2.7.1 $\mathbb{R}$ is homeomorphic to $(-1, 1)$.

Note that $(-1, 1) \subseteq \mathbb{R}$ and it inherits the subspace topology.

Proof. Choose $f : (-1, 1) \to \mathbb{R}$ where $f(x) = \frac{x}{1-x^2}$.

f is injective (i.e., $\forall x_1, x_2 \in (0, 1), f(x_1) = f(x_2) \Rightarrow x_1 = x_2$).

f is surjective (i.e., $\forall y \in \mathbb{R}, \exists x \in (-1, 1), f(x) = y$).

We can show f is continuous topologically by showing it is continuous analytically (which is equivalent in a subspace of $\mathbb{R}$ and its Euclidean topology). Since f is a rational function, it is continuous on $\text{dom}(f) = \mathbb{R} - \{\pm 1\}$. Observe $(-1, 1) \subseteq \mathbb{R} - \{\pm 1\}$, so f is continuous on $(-1, 1)$.

Now we need to show f^{-1} is continuous. We find $f^{-1} = \frac{2x}{1+(1+4x^2)^{\frac{1}{2}}}$, which is continuous. (Note that $1 + (1 + 4x^2)^{\frac{1}{2}} > 0$ for all $x \in \mathbb{R}$, so $f^{-1}(x)$ is continuous in its domain). □

We could also choose $g(x) = \tan\left(\frac{\pi x}{2}\right)$ to show $(-1, 1) \cong \mathbb{R}$.

Exercise 2.9 Let $a, b \in \mathbb{R}, a < b$. Prove $(a, b) \cong \mathbb{R}$.

■ **Example 2.22** Consider $S^2 - \{0, 0, 1\}$, which is the sphere without its north pole (since it is a subset of $\mathbb{R}^3$, its topology is the subspace topology of the standard topology of $\mathbb{R}^3$). Then $S^2 - \{0, 0, 1\} \cong \mathbb{R}^2$ by considering the stereographic projection. ■

2.7.1 Topological invariants

How do we show two topological spaces are not homeomorphic?

Definition 2.7.2 — Topological invariant. A *topological invariant* is a property that is preserved under homeomorphism.

■ **Example 2.23** Which of the below are topological invariants (when these properties are defined)?

- Boundedness is not a topological invariant, for example, Proposition 2.7.1.
- Connectedness is a topological invariant.
- Cardinality is a topological invariant because a homeomorphism is a bijection.

- Area is not a topological invariant, for example, Example 2.22. ■

2.8 Closure and Interior

Recall a closed set is a set whose complement is open. Also, arbitrary intersection and finite union of closed sets are closed.

Definition 2.8.1 — Closure and interior of a set. [Mun99, page 95] Let X be a topological space. Let $A \subseteq X$.

The *closure* of A, $\overline{A}$, is the intersection of all closed subsets of X that contain A.

The *interior* of A, $\mathring{A}$, is the union of all open subsets of X contained in A.

The closure is equivalent to the smallest closed set containing A, and the interior is equivalent to the largest open set contained in A.

Proposition 2.8.1 Let X be a topological space. Let $A \subseteq X$. Then

(i) $\mathring{A}$ is open in X.

(ii) $\overline{A}$ is closed in X.

(iii) A is closed iff $\overline{A} = A$.

(iv) A is open iff $\mathring{A} = A$.

(v) $\mathring{A} \subseteq A \subseteq \overline{A}$.

We don't use the interior as often as the closure. The following will help us define continuity at a point.

Proposition 2.8.2 Let X be a topological space. Let $A \subseteq X$.
$x \in \mathring{A}$ iff $\exists U \in \mathcal{T}_X$ such that $x \in U \subseteq A$.

We call this $x \in \mathring{A}$ an *interior point*.

Proof. Suppose $x \in \mathring{A}$.
Choose $U = \mathring{A} \in \mathcal{T}_X$ by Proposition 2.8.1(i).
Then $x \in \mathring{A}$ by supposition, and $U = \mathring{A} \subseteq A$ by Proposition 2.8.1(ii).
$\therefore \exists U \in \mathcal{T}_X$ such that $x \in U \subseteq A$.
Suppose $\exists U \in \mathcal{T}_X$ such that $x \in U \subseteq \mathring{A}$.
Then $U \subseteq \mathring{A}$ by definition of $\mathring{A}$.
$\therefore x \in \mathring{A}$. □

Proposition 2.8.3 Let X be a topological space. Let $A \subseteq X$. Then $X - \overline{A} = (\mathring{X - A})$.

Proposition 2.8.4 Let X be a topological space. Let $A \subseteq X$. Then $x \notin \overline{A}$ iff $\exists U \in \mathcal{T}_X$ such that $x \in U$ and $U \cap A = \varnothing$.

Proof. Suppose $x \notin \overline{A}$.
Then $x \in X - \overline{A} = (\mathring{X - A})$ by Proposition 2.8.3.
Choose $U = (\mathring{X - A})$. Then $U \in \mathcal{T}_X$ by Proposition 2.8.1(i), and $x \in U$. Also $U \cap A = \varnothing$ because $A \cap (X - A) = \varnothing$ and $U \subseteq (X - A)$.
$\therefore \exists U \in \mathcal{T}_X$ such that $x \in U$ and $U \cap A = \varnothing$. □

Theorem 2.8.5 [Mun99, page 95] Let X be a topological space. Let $Y \subseteq X$ be a subspace. Let $A \subseteq Y$ be a subset. Let $\overline{A}$ be the closure of A in X.
Then the closure of A in Y is $\overline{A} \cap Y$.

Recall B is closed in Y iff $B = C \cap Y$ for some closed C in X.
We want to show the closure of A in Y is $B = \overline{A} \cap Y$ by opposite set containment.

Theorem 2.8.6 — Basis Closure Theorem. [Mun99, page 96] Let X be a topological space. Let $A \subset X$ be a subset. Let β be a basis for $\mathcal{T}_X$.
(a) $x \in \overline{A}$ iff $\forall U \in \mathcal{T}_X (x \in U \Rightarrow U \cap A \neq \varnothing)$. That is, every open set U in X with $x \in U$ intersects A.
(b) $x \in \overline{A}$ iff $\forall B \in \beta (x \in B \Rightarrow B \cap A \neq \varnothing)$. That is, every basis element B with $x \in B$ intersects A.

Proof. (a) Logically equivalent to Proposition 2.8.4 by contrapositive.
(b) Suppose $x \in \overline{A}$. By (a), every open set of U of X with $x \in U$ intersects A. Consider $U = \cup_i B_i$ for some $B_i \in \beta$. Then every basis element B_j with $x \in B_j$ intersects A.
Suppose that every basis element B_j with $x \in B_j$ intersects A. Then so does every open V in X with $x \in V$ because $x \in B_j \subseteq V$ by definition of β generates $\mathcal{T}_X$.
□

Theorem 2.8.7 Let X be a topological space and let $Y \subseteq X$ be a subspace. Let $A \subseteq Y$ be a subset. Suppose $\overline{A}$ is the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Now, one of the possibly advantageous things that these ‘elementary concepts provide us with is an equivalent way to check that a function is continuous.

2.8.1 Limit Points

Definition 2.8.2 — Limit point. A point $x \in X$ is a *limit (or accumulation, or cluster) point* iff $\forall U \in \mathcal{T}_X, (U - \{x\}) \cap A \neq \varnothing$, i.e. any deleted neighborhood around x intersects A.

Notation 2.1. *The set of all limit points of A is* A'

Proposition 2.8.8 $\overline{A} = A \cup A'$.

2.8.2 Continuity at a point and continuity equivalents

Theorem 2.8.9 Let X and Y be topological spaces. Let $f : X \to Y$ be a function. Then the following are equivalent:
1. f is continuous
2. Every closed set in Y has its pre-image closed in X.
3. $\forall x \in X$ and $V \in \mathcal{T}_Y$ with $f(x) \in V$, $\exists U \in \mathcal{T}_X$ such that $x \in U$ and $f(U) \subseteq V$. (If this holds for a single point, we get the following definition for continuity at a point.)
4. For all subsets $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. That is, the image of the closure of A is contained in the closure of the image of A.

Proof of $1 \Rightarrow 4$. Suppose $f : X \to Y$ is continuous. Let $A \subseteq X$ be an arbitrary subset.
We want to show $f(\overline{A}) \subseteq \overline{f(A)}$. That is, if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.
Let $x \in \overline{A}$ be arbitrary.
Let $V \in \mathcal{T}_Y$ with $f(x) \in V$. (Otherwise, if $f(x) \notin V$, the statement is vacuously true.)
Then $f^{-1}(V) \in \mathcal{T}_X$ because f continuous and $x \in f^{-1}(V)$ by definition of pre-image.
So $f^{-1}(V)$ by Basis Closure Theorem (Theorem 2.8.6(b)).

Then $f(f^{-1}(V) \cap A) \subseteq f(f^{-1}(V)) \cap f(A) = V \cap f(A)$ by set theory. (Each set is non-empty since f a function.)

$\therefore$ By Theorem 2.8.6, $f(x) \in \overline{f(A)}$. □

Definition 2.8.3 — Continuity at a point. Let X and Y be topological spaces. Let $f : X \to Y$ be a function.

We say f is continuous at x iff $\forall V \in \mathcal{T}_Y$ such $f(x) \in V$, $\exists U \in \mathcal{T}_X$ such that $x \in U$ and $f(U) \subseteq V$.

2.8.3 Continuity at a point

Definition 2.8.4 — Continuity at a point. Let X, Y be topological spaces. Let $f : X \to Y$.

Then f is *continuous at a point* $a \in X$ iff $\forall V \in \mathcal{T}_Y$ such that $f(a) \in V$, $\exists U \in \mathcal{T}_X$ such that $a \in U$ and $f(U) \subseteq V$.

2.9 Metric Spaces

Definition 2.9.1 — Metric Space. [Mun99, page 119] Let $X \neq \varnothing$. A *metric space* (X, d) where $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function (a *metric function*) such that:

(i) $d(x,y) \geq 0$ for all $x, y \in X$ (non-negativity)
(ii) $d(x,y) = 0$ iff $x = y$
(iii) $d(x,y) = d(y,x)$ for all $x, y \in X$ (symmetry)
(iv) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$ (triangle inequality)

Notation 2.2. *The number $d(x,y)$ is called the distance between x and y.*

■ **Example 2.24** (a) Let $X = \mathbb{R}^n$ and $d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^n (x_i, y_i)^2\right]^{1/2}$. Then (X,d) is a metric space, called the standard Euclidean metric.

(b) Let $X = C([0,1]) = \{f \mid f : [0,1] \to \mathbb{R}, f \text{ continuous}\}$ and

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$

Then (X,d) is a metric space, where d represents the difference in the area of the curves of f and g.

(c) Let $X = C([0,1])$ and $d(f,g) = \sup_{x \in [0,1]} \{|f(x) - g(x)|\}$. Then (X,d) is a metric space. We know d exists by completeness axiom of $\mathbb{R}$, and this d represents the biggest pointwise distance between the two functions.

■

Definition 2.9.2 — Trivial metric. Let $X \neq \varnothing$. Then

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

is the *trivial metric*.

Definition 2.9.3 — ε-neighbourhood. Let (X,d) be a metric space. Let $x \in X, \varepsilon > 0$.
Then the ε-neighbourhood is

$$N_\varepsilon(x) = B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$$

For example, we have the open balls in $\mathbb{R}^2$ as ε-neighbourhood.

2.9.1 Metric spaces and topological spaces

Definition 2.9.4 — Metric topology induced by d. [Mun99, page 119] Let (X,d) be a metric space. The set

$$\beta = \{B_d(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$$

forms a basis for a topology $\mathcal{T}_d$ on X. The topology $\mathcal{T}_d$ is the *metric topology induced by d*.

■ **Example 2.25** Let $X \neq \varnothing$. Let d be the trivial metric.
Let $x \in X$. Consider $B_d(x,\frac{1}{2}) = \{y \in X \mid d(x,y) < \frac{1}{2}\} = \{x\}$. So $\forall x \in X, \{x\} \in \mathcal{T}_d$.
$\therefore \mathcal{T}_d = \mathcal{T}_{\text{dis}}$. ■

We showed that it is always possible to turn a metric space into a topological space. Is the other way around possible? In general, no.

Definition 2.9.5 — Metrizable. A topological space $(X, \mathcal{T}_X)$ is *metrizable* if there exists a metric d such that $\mathcal{T}_d = \mathcal{T}_X$.

A metrizable space is then a set X with both a topology $\mathcal{T}_X$ and a metric d.
Not all topological spaces are metrizable. For example, $\mathbb{R}$ with the lower limit topology provides a counterexample. We want to categorize the metrizable topological spaces.

Definition 2.9.6 — Regular. A topological space $(X, \mathcal{T}_X)$ is regular if for all closed sets C of X and for all $x \notin C$, there exists $U, V \in \mathcal{T}_X$ such that $C \subseteq U$, $x \in V$, and $U \cap V = \varnothing$.

We can think of this as separating closed sets from points.

Theorem 2.9.1 — Urysohn's Metrization Theorem. [Mun99, page 212] Every regular topological space with a countable basis is metrizable.

■ **Example 2.26** The Euclidean space $\mathbb{R}^n$ with the standard topology is regular and has a countable basis. It is indeed metrizable. ■

2.9.2 Convergence

Definition 2.9.7 — Open set in a metrizable space. Let X be a metrizable space. A set U is *open* in X if $\forall x \in U$, $\exists B_d(x,\varepsilon)$ such that $x \in B_d(x,\varepsilon) \subseteq U$.

Definition 2.9.8 — Convergence in a metrizable space. [Mun99, page 129] Let X be a metrizable space. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$.
We say that $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$ iff $\exists x \in X$, $\forall \varepsilon > 0$, $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $n > M \Rightarrow x_n \in B_d(x,\varepsilon)$.

That is, for any size neighbourhood around the limit, the tail end of the sequence is in the neighbourhood.

For example, if $X = \mathbb{R}$, this is just the δ-M definition of convergence from calculus.

Definition 2.9.9 — Convergence in a topological space. [Mun99, page 89] Let X be a topological space. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$.

We say that $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$ iff $\exists x \in X$, $\forall U \in \mathcal{T}_x$ such that $x \in U$, $\exists M > 0$, $\forall n \in \mathbb{N}$, $n > M \Rightarrow x_n \in U$.

That is, the tail end of the sequence is in every open set covering the limit.

Note that convergence is not necessarily unique in all topological spaces (for example, consider the trivial topology), but it is unique in metrizable spaces.

Exercise 2.10 Let $X = \mathbb{R}$ with $\mathcal{T}_{\text{std}}$. Note $(\mathbb{R}, \mathcal{T}_{\text{std}})$ is metrizable. Prove $\{\frac{n!}{(n+2)!}\}$ converges to 0 in X.

Exercise 2.11 Let X be metrizable. Let $\{x_n\} \subseteq X$, $\bar{x}, \bar{y} \in X$.

Prove if $x_n \to \bar{x}$ (i.e. x_n converges to $\bar{x}$) and $x_n \to \bar{y}$, then $\bar{x} = \bar{y}$.

We can prove X is metrizable by using Urysohn's Metrization Theorem or by definition (i.e., show there is a metric such that the metric topology induced by the metric is equivalent to the topology).

2.9.3 Continuity in a metrizable space

Definition 2.9.10 — Continuity in a metrizable space. [Mun99, page 129] Let X, Y be metrizable spaces with metrics d_X, d_Y respectively. Let $f : X \to Y$.

We say f is *continuous at a point* $a \in X$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

Recall continuity at a point for a general topological space (Definition 2.8.4). Then we get an equivalent statement: the image of the delta neighborhood at a point is a subset of the epsilon neighborhood at the image of the point, i.e. $f(N_\delta(a)) \subseteq N_\varepsilon(f(a))$. Then, we can consider open sets $U = f(N_\delta(a))$ and $V = N_\varepsilon(f(a))$.

Theorem 2.9.2 [Mun99, page 129] Let X, Y be metrizable spaces with metrics d_X, d_Y respectively. Let $f : X \to Y$.

Then f is continuous iff $\forall x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

Again, we will use the fact that the δ-ε condition is equivalent to $f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$.

Proof. Suppose f is continuous.

Let $x \in X$ be arbitrary. Let $\varepsilon > 0$ be arbitrary.

Consider $f^{-1}(N_\varepsilon(f(x)))$, which is open in X because $N_\varepsilon(f(x))$ is open in Y and f is continuous. Also $x \in f^{-1}(N_\varepsilon(f(x)))$ by definition of pre-image.

Since $f^{-1}(N_\varepsilon(f(x)))$ is open, we can choose $\delta > 0$ such that $N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x)))$.

Applying f to both sides of the set containment $N_\delta(x) \subseteq f^{-1}(N_\varepsilon(f(x)))$, we get $\exists \delta > 0$ such that $f(N_\delta(x)) \subseteq N_\varepsilon(f(x))$

$\therefore \forall x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Suppose δ-ε condition.

Let $V \in \mathcal{T}_Y$ be arbitrary.

Consider $f^{-1}(V)$, which we want to show is open.
Let $x \in f^{-1}(V)$ be arbitrary.
We know that $f(x) \in V$ by definition of pre-image and V open in Y.
Since V is open (in a metrizable space), $\exists \varepsilon > 0$ such that $N_\varepsilon(f(x)) \subseteq V$.
By δ-ε condition, $\exists \delta > 0$ such that $N_\delta(x) \subseteq f(N_\varepsilon(f(x))) \subseteq V$.
So $x \in \underbrace{N_\delta(x)}_{\subseteq f^{-1}(f(N_\delta(x)))} \subseteq f^{-1}(N_\delta(f(x))) \subseteq f^{-1}(V)$. $N_\delta(x) \subseteq f^{-1}(V)$ is the definition of $f^{-1}(V)$ being open in a metrizable space.
$\therefore f$ is continuous. □

Theorem 2.9.3 — Continuity by convergence. [Mun99, page 130] Let X, Y be metrizable spaces with metrics d_X, d_Y respectively. Let $f : X \to Y$.
f is continuous iff $\forall \{x_n\} \subseteq X$, if $x_n \to \bar{x}$ in X then $f(x_n) \to f(\bar{x})$ in Y.

Proof outline. Suppose f is continuous.
$\therefore$ Sequence condition holds.
Suppose sequence condition.
Proof by contradiction, i.e., for contradiction, assume f is not continuous.
$\therefore f$ is continuous. □

2.10 Quotient Spaces

We want a general procedure to build a complicated topological space from a given topological space. For example, given some topological spaces, we can glue them into new topological spaces.
Given topological space X and a map $f : X \to Y$. Can we describe a topological space on Y? Yes.

Definition 2.10.1 — Quotient space. [Mun99, page 137] Let X be a topological space. Let $Y \neq \varnothing$.
The set $(Y, \mathcal{T}')$ is called the *quotient space* if there exists a surjective map $p : X \to Y$ such that $\forall U \subseteq Y$, $U \in \mathcal{T}'$ iff $p^{-1}(U) \in \mathcal{T}_X$.
p is a *quotient map* and $\mathcal{T}'$ is the *quotient topology*.

It follows by definition that $p : X \to Y$ is continuous.
Let X, Y be topological spaces. If $f : X \to Y$ is surjective, continuous, and open, then f is a quotient map. Also, if $f : X \to Y$ is surjective, continuous, and closed (i.e. the image of closed sets is closed), then f is a quotient map.

Proof $\mathcal{T}'$ is a topology.
(a) $Y \subseteq Y$. $p^{-1}(Y) = X \in \mathcal{T}_X$ (since p surjective). $\therefore Y \in \mathcal{T}'$.
$\varnothing \subseteq Y$. $p^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$. $\therefore \varnothing \in \mathcal{T}'$.
(b) Let $\{U_\alpha\}_{\alpha \in J}$ be any collection of $U_\alpha \in \mathcal{T}'$.
Consider $\bigcup_{\alpha \in J} U_\alpha$.
$p^{-1}(\bigcup_{\alpha \in J} U_\alpha) = \bigcup_{\alpha \in J} \underbrace{p^{-1}(U_\alpha)}_{\in \mathcal{T}_X}$ is open in X since X closed under arbitrary union.
$\therefore \bigcup_{\alpha \in J} U_\alpha$ is open in Y.
(c) By induction, it is sufficient to consider pairwise intersection.
Let $U, V \in \mathcal{T}'$.
Consider $U \cap V$.
$p^{-1}(U \cap V) = \underbrace{p^{-1}(U)}_{\in \mathcal{T}_X} \cap \underbrace{p^{-1}(V)}_{\in \mathcal{T}_X}$ is open in X since X closed under finite intersection.
$\therefore U \cap V$ is open in Y.

□

■ **Example 2.27** Let $X = \mathbb{R}$, $Y = \{a,b,c\}$. Let $p : X \to Y$ be a quotient map defined by

$$p(x) = \begin{cases} a, & \text{if } x > 0 \\ b, & \text{if } x < 0 \\ c, & \text{if } x = 0 \end{cases}$$

We consider all $2^3 = 8$ subsets of Y, compute their pre-image, and check if they are open in X.

- $\varnothing \subseteq Y$. $p^{-1}(\varnothing) = \varnothing \in \mathcal{T}_{\text{std}}$. $\therefore \varnothing \in \mathcal{T}'$.
- $Y \subseteq Y$. $p^{-1}(Y) = \mathbb{R} \in \mathcal{T}_{\text{std}}$. $\therefore Y \in \mathcal{T}'$.
- $\{a\} \subseteq Y$. $p^{-1}(\{a\}) = (0,\infty) = \bigcup_{n\in\mathbb{N}}(0,n) \in \mathcal{T}_{\text{std}}$. $\therefore \{a\} \in \mathcal{T}'$
- $\{b\} \subseteq Y$. $p^{-1}(\{b\}) = \underbrace{(-\infty,0)}_{=\bigcup_{n\in\mathbb{N}}(-n,0)} \in \mathcal{T}_{\text{std}}$. $\therefore \{b\} \in \mathcal{T}'$
- $\{c\} \subseteq Y$. $p^{-1}(\{c\}) = \{0\} \notin \mathcal{T}_{\text{std}}$ because $\mathbb{R} - \{0\} = (-\infty,0) \cup (0,\infty)$ is open, so $\{0\}$ is closed, and the only clopen sets in $\mathbb{R}$ are $\varnothing, \mathbb{R}$. $\therefore \{c\} \notin \mathcal{T}'$.
- $\{a,b\} \subseteq Y$. $p^{-1}(\{a,b\}) = (-\infty,0) \cup (0,\infty) \in\in \mathcal{T}_{\text{std}}$. $\therefore \{a,b\} \in \mathcal{T}'$.
- $\{b,c\} \subseteq Y$. $p^{-1}(\{b,c\}) = [0,\infty) \notin \mathcal{T}_{\text{std}}$. $\therefore \{a,c\} \notin \mathcal{T}'$.
- $\{a,c\} \subseteq Y$. $p^{-1}(\{a,c\}) = (-\infty,0] \notin \mathcal{T}_{\text{std}}$. $\therefore \{b,c\} \notin \mathcal{T}'$.

■

Recall an equivalence relation is a reflexive, symmetric, transitive binary relation which creates a partition of a set.

Definition 2.10.2 — Quotient space . [Mun99, page 139] Let X be a topological space. Let $\sim$ be an equivalence relation.

Let $[x] = \{y \in X \mid y \sim x\}$ be the equivalence class of $x \in X$.

The *quotient space* $X^* = X/\sim = \{[x] \mid x \in X\}$ equipped with the topology $\mathcal{T}'$ such that $\forall U \subseteq X^*$,

$$U \in \mathcal{T}' \text{ iff } p^{-1}(U) \in \mathcal{T}_X$$

where $p : X \to X/\sim$ is the (canonical) projection map $x \mapsto [x]$.

We also call this space the *identification space of X under* $\sim$.

That is, the pre-image (with respect to the projection map) of open sets in the identification space are open sets in the original topology.

■ **Example 2.28** Realize $T = S^1 \times S^1$ as a quotient space of a rectangle $X = [0,1] \times [0,1]$ by specifying:

(a) an appropriate edge identification diagram.
(b) an equivalence relation on X.

■

Figure 2.1 gives the edge identification diagram for (a).

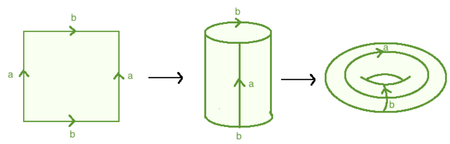

Figure 2.1: Edge identification diagram of X

For part (b), consider $X = \{x \times y \in X \mid x \in [0,1], y \in [0,1]\}$. We get the following equivalence relations:

- For each $t \in (0,1)$, $(0,t) \sim (1,t)$.
- For each $s \in (0,1)$, $(s,0) \sim (s,1)$.
- $(0,0) \sim (0,1) \sim (1,1) \sim (1,0)$.
- For each $x \in X$, $x \sim x$.

Exercise 2.12 Let $X = [0,1]$ and let $\sim$ be the relation on X such that $0 \sim 1$ and $x \sim x$ for all $x \in (0,1)$. What is X^* (up to homeomorphism)?

3. Connectedness and Compactness

3.1 Connected Spaces

The following notion provides another topological invariant.

Definition 3.1.1 — Connected spaces. [Mun99, page 148] Let X be a topological space.

X is *connected* iff $\nexists U,V \in \mathcal{T}_X$ such that $U,V \neq \varnothing$ and $X = U \sqcup V$ (i.e., X is the disjoint union of U and V).

X is *disconnected* iff it is not connected, i.e., $\exists U,V \in \mathcal{T}_X$, $U,V \neq \varnothing$, $X = U \sqcup V$. If such a pair exists, we call U,V a *separation* of X.

■ **Example 3.1** Let X be a topological equipped with $\mathcal{T}_{\text{dis}}$. Suppose $|X| > 1$. Then X is disconnected. ■

Proof. We know $\mathcal{T}_{\text{dis}} = \mathcal{P}(X)$. Choose $U = \{x\} \in \mathcal{T}_{\text{dis}}$ and $V = X - \{x\} \in \mathcal{T}_{\text{dis}}$. Since $|X| > 1$, $U,V \neq \varnothing$ and $U,V \subset X$. By construction $U \cap V = \varnothing$, $X = U \sqcup V$.

$\therefore X$ is disconnected. □

Theorem 3.1.1 [Mun99, page 148] Let X be a topological space. X is connected iff the only clopen sets of X are $\varnothing$ and X.

Proof. For contrapositive of $(\Rightarrow)$, suppose there exists a clopen set $A \subset X$, $A \neq \varnothing$. We want to show X is disconnected.

Then choose $U = A \in \mathcal{T}_X$ because A is clopen (open). Choose $V = X \setminus A \in \mathcal{T}_X$ because A is clopen (closed).

$U \neq \varnothing$ by hypothesis and $V \neq \varnothing$ because $A \subset X$. Also $U \cap V = \varnothing$. Moreover, $X = A \sqcup (X \setminus A) = U \sqcup V$.

$\therefore X$ is disconnected.

For contrapositive of $(\Leftarrow)$, suppose X is disconnected. We want to show there exists a clopen set $A \subset X$, $A \neq \varnothing$.

Since X is disconnected, there exists a separation $U,V \in \mathcal{T}_X$ such that $U,V \neq \varnothing$, $U,V \subset X$,

$U \cap V = \varnothing$ and $X = U \cup V$ (i.e., $X = X \sqcup V$).

Choose $A = U \in \mathcal{T}_X$. Then $A \neq \varnothing$ and $A \subset X$ by hypothesis. Also $X - A = X - U = V$ because $X = U \sqcup V$ by hypothesis.

$\therefore$ A is clopen. □

Example 3.2 (a) $(\mathbb{R}, \mathcal{T}_{\text{std}})$ is connected. We've previously shown the only clopen sets are $\mathbb{R}$ and $\varnothing$.

(b) $\mathbb{R}_\ell$ is disconnected. Then $\mathcal{T}_\ell$ is the topology generated by by basis

$$\beta = \{\, [a,b) \mid a,b \in \mathbb{R},\, a < b \,\}.$$

■

Proof of (b). Consider $[a,b) \in \mathcal{T}_X$ because $\beta \subset \mathcal{T}$.

Now consider $\mathbb{R} \setminus [a,b) = (-\infty, a) \cup [b, \infty)$.

We have $\cup_{n \in \mathbb{N}} \big([-n + a, a) \cup [b, b+n) \big) = (-\infty, a) \cup [b, \infty)$ is open as the union of basis elements. Then $\mathbb{R} \setminus [a,b)$ is closed.

Then $[a,b)$ is clopen, $[a,b) \neq \mathbb{R}, \varnothing$.

$\therefore \mathbb{R}_\ell$ is disconnected. □

Theorem 3.1.2 — Subspace connectedness lemma. [Mun99, page 148] Let X be a topological space. Let $Y \subseteq X$ be a subspace of X.

Then Y is disconnected iff there exists $A, B \subseteq Y$ such that $A, B \neq \varnothing$, $A \cup B = Y$, and A does not contain any limit points of B and B does not contain any limit points of A.

The limit points are with respect to X.

Recall a point $x \in X$ is a limit point iff for all open sets U, if $x \in U$, then $(U \setminus \{x\}) \cap A \neq \varnothing$. Note these A, B are not necessarily open and so this pair A, B is not necessarily a separation.

Proof of $(\Rightarrow)$. Suppose Y is disconnected. Then there exists a separation A, B of Y.

We already have that $A, B \subseteq Y$ such that $A, B \neq \varnothing$, $A \cup B = Y$. We only need to show the limit point condition.

Since $Y = A \sqcup B$ and A, B open, A and B are closed in Y. Then A is equal to the closure of A in Y,

$$A = \underbrace{\overline{A}}_{\text{closure in } X} \cap Y$$

and B is equal to the closure of B in Y,

$$B = \underbrace{\overline{B}}_{\text{closure in } X} \cap Y$$

by Theorem 2.8.5.

Recall $\overline{A} = A \cup A'$ (the set of limit points of A in X).

We have the following:

$$
\begin{aligned}
& A \cap B = \varnothing \\
\Rightarrow\ & (\overline{A} \cap Y) \cap B = \varnothing \\
\Rightarrow\ & \overline{A} \cap B = \varnothing && (B \subseteq Y) \\
\Rightarrow\ & (\overline{A} \cup A') \cap B \\
\Rightarrow\ & (A \cap B) \cup (A' \cap B) = \varnothing \\
\Rightarrow\ & \varnothing \cup (A' \cap B) = \varnothing && (A, B \text{ separated}) \\
\Rightarrow\ & A' \cap B = \varnothing
\end{aligned}
$$

$\therefore$ B does not contain any limit points of A.
We have analogous argument by switching A and B.
$\therefore$ A does not contain any limit points of B. □

Proposition 3.1.3 — Connectedness preserved under continuity. [Mun99, page 150] Let X, Y be topological spaces. Let $f : X \to Y$ be continuous.
If X is connected, then the image $f(X)$ is connected in Y.

Note that the image $f(X)$ is a subspace of Y.

Proof. Let X, Y be a topological space.
Let $f : X \to Y$ be connected.
Suppose X is connected.
For contradiction, assume that $f(X)$ is disconnected.
Then there exists a separation U, V of $f(X)$. So $f(X) = U \sqcup V$.
Note that since f is continuous, $f : X \to f(X)$ is continuous (i.e., we can restrict the codomain to the range).
Since U, V are open in $f(X)$, the preimage of $f(X)$ is $X = f^{-1}(U \sqcup V) = f^{-1}(U) \sqcup f^{-1}(V)$, which are open (by continuity on the restricted map f) and nonempty, and so they form a separation of X. Then X is disconnected. Contradiction.
$\therefore$ $f(X)$ is connected in Y. □

The Intermediate Value Theorem is just a special case of this theorem. Let $X = [a, b]$ and $Y = \mathbb{R}$.

Proposition 3.1.4 — Connectedness preserved under finite product. $X_1 \times X_2 \times \cdots \times X_n$ is connected when X_i is connected for all $i = 1, \ldots, n$.

Proposition 3.1.5 [Mun99, page 150] Let X be a topological space.
Let $A \subset X$ be a connected subspace.
If $\exists B \subseteq X$ subspace such that $A \subset B \subseteq \overline{A}$, then B is connected.

Said differently: If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.

Check the proof in the textbook for reference.

Proof. Let $A \subset X$ be a connected subspace.
Suppose $\exists B \subseteq X$ subspace such that $A \subset B \subseteq \overline{A}$.
For contradiction, assume B is disconnected.
Consider a seperation U, V of B. That is, $\exists U, V \subseteq B$, $U, V \neq \varnothing$, U, V are open in B, $U \cap V = \varnothing$ and $B = U \cup V$.
Since U, V open in B, then $\exists U_X, V_X \in \mathcal{T}_X$, $U_X, V_X \neq \varnothing$ such that $U = B \cap U_X$ and $V = B \cap V_X$.

Since $U \cap V = \varnothing$, then $U_X \cap V_X \cap B = \varnothing$.

Since $B = U \cup V$, then $B \subseteq U_X \cup V_X$. So $A \subset U_X \cup V_X$ and $U_X \cap V_X \cap A = \varnothing$. We can refine this to $U_X \cap A = \varnothing$ or $V_X \cap A = \varnothing$ because A is connected (if neither are true, then there would exist a separation).

WLOG, assume $U_X \cap A = \varnothing$.

Recall $U = B \cap U_X \neq \varnothing$ and $B \subseteq \overline{A}$. Then

$$\exists x \in B \cap U_X \subseteq \overline{A} \cap U_X = (A \cup A') \cap U_X = (A \cap U_X) \cup (A' \cap U_X) = \varnothing A' \cap U_X.$$

So $x \in A'$ and $x \in B$. Contradiction to Theorem 3.1.2.

$\therefore B$ is connected. □

Exercise 3.1 Prove that the only proper subspaces of $\mathbb{R}$ that are connected are $\{c\}$, (a,b), $[a,b)$, $(a,b]$ and $[a,b]$ where $a,b,c \in \mathbb{R}$, $a < b$.

3.2 Path Connectedness

Definition 3.2.1 — Path. [Mun99, page 155] Let X be a topological space. Let $x,y \in X$.

A *path* from x to y (or an xy-path) is a continuous function $f : [a,b] \to X$ for some $a,b \in \mathbb{R}$ such that $f(a) = x$ and $f(b) = y$.

Definition 3.2.2 — Path-connected. [Mun99, page 155] Let X be a topological space.

X is *path-connected* iff for all $x,y \in X$, there exists an xy-path in X.

■ **Example 3.3** $X = \mathbb{R}^n$ is path-connected.

Let $a,b \in \mathbb{R}^n$.

Define $f : [0,1] \to \mathbb{R}^n$ by $f(t) = (1-t)a + bt$. We need to check f is continuous, $f(0) = a$, and $f(1) = b$. (We understand $\mathbb{R}^n$ and know that this f is continuous already.) ■

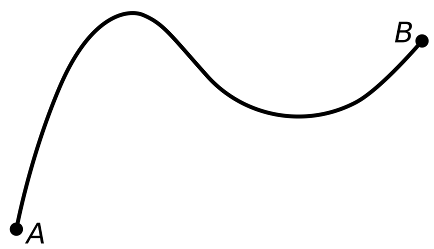

Figure 3.1: Example of a path

■ **Example 3.4** Let $(X, \mathcal{T}_{\text{trivial}})$ (where $|X|$ is at most countable). Then X is path-connected.

Let $x,y \in X$.

Consider any surjective $f : [0,1] \to X$ such that $f(0) = x$, $f(1) = y$. Check continuity. Recall $\mathcal{T}_{\text{trivial}} = \varnothing$.

We get $f(\varnothing) = \varnothing = \varnothing \cap [0,1] \in \mathcal{T}_{[0,1]}$. Also, since f surjective, $f(X) = [0,1] = [0,1] \cap \mathbb{R}$. Then f is continuous.

Exercise: Does such a map f exist? ■

Proposition 3.2.1 [Mun99, page 155] Let X be a topological space.
If X is path-connected, then X is connected.

Proof. Suppose X is path-connected.

For contradicition, assume that X is disconnected. That is, there exists $U, V \in \mathcal{T}_X$ such that $U, V \neq \varnothing$ and $U \sqcup V = X$.

Let $u \in U, v \in V$. Since X is path-connected, there exists a continuous function $f : [a,b] \to X$ such that $f(a) = u$ and $f(b) = v$.

Since $[a,b]$ is connected, $f([a,b])$ is connected by Proposition 3.1.3. Then $f([a,b]) \subseteq U$ or $f([a,b]) \subseteq V$ (otherwise, $f([a,b])$ is disconnected).

WLOG, suppose $f([a,b]) \subseteq U$. Then $f(a), f(b) \in f([a,b])$, so $f(a) = u, f(b) = v \in U$. Contradiction. □

(R) The converse of Proposition 3.2.1 is false.

■ **Example 3.5 — The topologist's sine curve.** Define $f : (0,1] \to \mathbb{R}$ by $f(x) = \sin\left(\frac{1}{x}\right)$.

Let $S = f((0,1]) = \{\, (x, f(x)) \mid x \in (0,1] \,\}$, which is connected by Proposition 3.1.3 because $(0,1]$ is connected and f is continuous. (This is path-connected.)

Take $X = \overline{S} = X \cup \{(0,0)\}$, which is connected by Proposition 3.1.5.

Claim: X is not path-connected.

For contradiction, assume X is path-connected. Then any map from $(0,0)$ to some other point is not continuous [Mun99, page 157]. ■

■ **Example 3.6** Intervals in $\mathbb{R}$ are path-connected, so they are connected.
Circles in $\mathbb{R}^2$ are path-connected, and again they are connected. ■

Theorem 3.2.2 (a) Let X, Y be topological spaces. Let $f : X \to Y$ be continuous.
If X is path-connected, then $f(X)$ is path-connected.
(b) Let $\{X_\alpha\}_{\alpha \in J}$ be topological spaces.
If each X_α is path-connected, then $\prod_{\alpha \in J} X_\alpha$ is path-connected (in the product topology).

However, Theorem 3.1.5 does not hold for path-connectedness.

3.3 Compactness

Definition 3.3.1 — Covering, Open Covering. [Mun99, page 164] Let X be a topological space.

A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *covering* of X if the union of the elements of $\mathcal{A}$ equals X. $\mathcal{A}$ is an *open covering* of X if $\mathcal{A}$ is a covering and each element in $\mathcal{A}$ is open in X.

Definition 3.3.2 — Compact. Let X be a topological space.

X is *compact* iff every open covering of X has a finite subcovering (that is, a subset of A that is still a covering).

■ **Example 3.7** (i) $\mathbb{R}$ is not compact. Consider $\mathbb{R} = \cup_{n \in \mathbb{N}} (-n, n)$. There is no finite subcovering.
(ii) $(0,1)$ and $(0,1]$ are not compact.

■

Exercise 3.2 Let $(X,\mathcal{T})$ be a finite topological space. Suppose $|X| < \infty$. Prove that $(X,\mathcal{T})$ is compact.

Theorem 3.3.1 — Subspace compactness lemma. [Mun99, page 164] Let X be a topological space. Let $Y \subseteq X$ be a subspace.

Then Y is compact iff, if $Y \subseteq \bigcup_{\alpha \in J} U\alpha$ where $U_\alpha \in \mathcal{T}_X$, then there exists $\alpha_1, \ldots, \alpha_n \in J$ such that $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

That is, given U_α is an covering of Y with sets open in Y, we have a finite subcovering of Y.

Proof. Suppose Y is compact.

Suppose $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$ where $U_\alpha \in \mathcal{T}_X$.

$$Y = Y \cap \bigcup_{\alpha \in J} U_\alpha = \bigcup_{\alpha \in J} (Y \cap U_\alpha)$$

The collection $\{U_\alpha \cap Y \mid \alpha \in J\}$ is an open covering of Y.

$\therefore$ Since Y is compact, there exists $\alpha_1, \ldots, \alpha_n \in J$ such that $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

Suppose that given U_α is an covering of Y with sets open in Y, we have a finite subcovering of Y.

Let $Y = \bigcup_{\alpha \in J} V_\alpha$ for any $V_\alpha \in \mathcal{T}_Y$.

Then $V_\alpha = Y \cap U_\alpha$ for some $U_\alpha \in \mathcal{T}_X$.

So $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$. By the hypothesis, there exists $\alpha_1, \ldots, \alpha_n \in J$ such that $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

Intersecting Y across, we get $Y = \bigcup_{i=1}^n (Y \cap U_{\alpha_i}) = \bigcup_{i=1}^n V_i$. This is a finite subcovering. □

Theorem 3.3.2 [Mun99, page 165] Let X be a compact topological space. Let $Y \subseteq X$ be a subspace.

If Y is closed, then Y is compact.

Proof. Suppose X is compact. Suppose $Y \subseteq X$ is a closed subspace.

Let $Y \subseteq \bigcup_{\alpha \in J} U_\alpha$ for any $U_\alpha \in \mathcal{T}_X$. Note this is a cover of Y open in X, not an open cover of X.

We have that $X = (\bigcup_{\alpha \in J} U_\alpha) \cup (X - Y)$ gives an open cover of X, since Y is closed. Then this cover has a finite subcovering $X = (\bigcup_{i=1}^n U_{\alpha_i}) \cup (X - Y)$.

Since $Y \subseteq X$, then $Y \subseteq (\bigcup_{i=1}^n U_{\alpha_i})$. By Lemma 3.3.1, Y is compact. □

Theorem 3.3.3 [Mun99, page 166] Let X, Y be topological spaces. Let X be compact.

If $f : X \to Y$ is continuous, then $f(X) \subseteq Y$ is compact.

Proof. Suppose X, Y topological spaces, X compact.

Suppose $f : X \to Y$ continuous.

We will apply Lemma 3.3.1 to $f(X)$ as a subspace of Y.

Let $\{U_\alpha\}_{\alpha \in J}$ be any cover of $f(X)$ by open sets in Y. That is, $f(X) \subseteq \bigcup_{\alpha \in J} U_\alpha$ for any $U_\alpha \in \mathcal{T}_Y$.

Taking pre-images, we get $X \subseteq f^{-1}(\bigcup_{\alpha \in J} U_\alpha) = \bigcup_{\alpha \in J} f^{-1}(U_\alpha)$. Since f is continuous, $f^{-1}(U_\alpha) \in \mathcal{T}_X$, giving us an open cover in X.

X is compact, so we can apply Lemma 3.3.1 to this open cover. Thus, there exists $\alpha_1, \ldots, \alpha_n \in J$ such that $X \subseteq \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$, a finite subcovering of X.

Taking the image preserves the containment, $f(X) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.
$\therefore$ By Lemma 3.3.1 on $f(X)$, $f(X)$ is compact. □

Then compactness is a topological invariant.

Theorem 3.3.4 [Mun99, page 167] Let $X_1, \ldots, X_n$ be a finite collection of compact topological spaces.
Then $X_1 \times \cdots \times X_n$ is compact.

We can prove the pairwise product is compact, and any finite product follows by induction.

Definition 3.3.3 — Hausdorff. [Mun99, page 1111111] A topological space X is *Hausdorff* or T_2 iff $\forall x_1, x_2 \in X$, $x_1 \neq x_2$, $\exists U, V \in \mathcal{T}_X$ such that $x_1 \in U$, $x_2 \in U$, and $U \cap V \neq \varnothing$.

Proposition 3.3.5 Every metric space $(X, \mathcal{T}_d)$ is Hausdorff.

Proof. Let $x, y \in X$, $x \neq y$.
Let $r = d(x,y)$. Then $r > 0$ because d is a metric and $x \neq y$.
Let $U = B_d(x, \frac{r}{4})$ and $V = B_d(x, \frac{r}{4})$. Then $U, V \in \mathcal{T}_d$ since they're basis elements.
For contradiction, suppose $U \cap V \neq \varnothing$. Then $\exists z \in U \cap V$.
By definition of intersection, $z \in U$ and $z \in V$. So $d(x,z) < \frac{r}{4}$ and $d(y,z) < \frac{r}{4}$.
By triangle inequality and symmetry, $r = d(x,y) \leq d(x,z) + d(y,z) = d(x,z) + d(z,y) = \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$. Since $r > 0$, contradiction.
$\therefore U \cap V =$, and so U, V are disjoint neighbourhoods of x, y.
$\therefore$ A metric space $(X, \mathcal{T}_d)$ is Hausdorff. □

Theorem 3.3.6 — Tychonoff's Theorem. If $\{X_\alpha\}_{\alpha \in J}$ is an arbitrary collection of compact topological spaces, then the product topology $\prod_{\alpha \in J} X_\alpha$ is compact.

Theorem 3.3.7 Let X be Hausdorff. Let $Y \subseteq X$. Suppose Y is compact. Then Y is closed.

Proof. Suppose X is Hausdorff and $Y \subseteq X$ is a compact subspace.
We show the set $X - Y$ is open by showing any point in it has a neighbourhood disjoint from Y, and thus contained in $X - Y$.
Let $x \in X - Y$.
For all $y \in Y$, there exists $U_y, V_y \in \tau_X$ such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y =$, which exists because τ_X is Hausdorff and $x \neq y$.
Note $Y \subseteq \bigcup_{y \in Y} V_y$. Then $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X, that is, $Y \subseteq \bigcup_{y \in Y} V_y$
.
Since Y is compact, we can use Lemma 3.3.1. There exists $y_1, \ldots, y_n \in Y$ such that $Y \subseteq \bigcup_{i=1}^{n} V_{y_i} = V$, i.e., we have a finite subcovering of Y by sets open in X.
Then the set $U = \bigcap_{i=1}^{n} U_{y_i}$ is open since it is the finite intersection of open sets, and the set V is open since it is the union of open sets.
We have $x \in U$ since $x \in U_y$ for all $y \in Y$ and $y \in V$ since $v \in V_y$ for all $y \in Y$.
For contradiction, suppose $U \cap V \neq \varnothing$.
Then $\exists z \in U \cap V$. So $z \in U_{y_i}$ for all y_i, $i = 1, \ldots, n$ and $z \in V_{y_i}$ for some y_i, $i = 1, \ldots, n$. But U_{y_i} and V_{y_i} disjoint by Hausdorff. Contradiction. So $U \cap V = \varnothing$.
That is, U_x is a neighbourhood of x disjoint from Y.
$\therefore X - Y = \bigcup_{x \in X - Y} U$ is the union of open sets, and is thus open itself. So Y is closed in X. □

These three theorems and their proofs are key:

1. Theorem 3.3.2 (Image of compact space is compact): Let X be a compact topological space. Let $Y \subseteq X$ be a subspace. If Y is closed, then Y is compact.
2. Theorem 3.3.3 (Closed subspace of a compact space is compact): Let X,Y be topological spaces. Let $f : X \to Y$ be continuous. If X is compact, then $f(X)$ is compact.
3. Theorem 3.3.7 (Compact subspace of a Hausdorff space is closed): Let X be a Hausdorff topological space. Let $Y \subseteq X$ be a subspace. If Y is compact, then Y is closed.

3.4 Compactness in Metric Spaces

Definition 3.4.1 — Bounded. Let (X,d) be a metric space. Then a set A is bounded if $\exists M \geq 0$ such that $d(x,y) \leq M$ for all $x,y \in A$.

Theorem 3.4.1 Let (X,d) be a metric space with the metric topology. Let $A \subseteq X$ be a nonempty subset.

If A is compact, then A is bounded.

Proof. Suppose $A \subseteq X$ is compact.

Then $A \subseteq \bigcup_{x\in X} B_d(x,\varepsilon_x)$ with $\varepsilon_x > 0$, which is an open covering of A by open sets in X. Since A is compact and by Lemma 3.3.1, $A \subseteq \bigcup_{i=1}^{n} B_d(x,\varepsilon_i)$.

Since we have finitely many open balls covering A, we can cover A with a single large enough open ball, i.e., $A \subseteq B_d(x_i,N)$, for some $M > 0$ and some x_i.

Then $\forall x,y \in A$, $d(x,y) \leq 2N$. □

This looks like one direction of Heine-Borel theorem.

Theorem 3.4.2 Let (X,d) be a metric space with the metric topology. Let $A \subseteq X$ be a subspace.

If A is compact, then A is closed and bounded.

Proof. By Proposition 3.3.5, metric spaces are Hausdorff. With Theorem 3.3.7, A is closed.

By Theorem 3.4.1, A is bounded. □

R The converse does not hold.

■ **Example 3.8 — Counterexample to converse of theorem 3.4.2.** Consider $\mathbb{R}$ with the trivial metric. Then $\mathbb{R}$ is closed and bounded in $\mathbb{R}$ (why?). But $\mathbb{R}$, which has the discrete topology as the metric topology, is not compact. ■

Theorem 3.4.3 — Heine-Borel Theorem. [Mun99, page 173] Let $X = \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be a subspace.

A is compact iff A is closed and bounded.

Proof. Suppose A is compact. $\mathbb{R}^n$ is metrizable with the Euclidean metric d.

$\therefore$ By Theorem 3.4.2, A is closed and bounded.

Suppose A is closed and bounded in d.

Let $x,y \in \mathbb{R}^n$, $x = (x_1,x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n)$.

We have another metric (which generates the same Euclidean topology), the square metric $\rho(x,y) = \max\{|x_i - y_i|\}$.

Since $\rho(x,y) \leq d(x,y)$, A is bounded in d implies A is bounded in ρ.

Let $x_0 \in A$. Then for any $x \in A$, $\rho(0,x) \leq \rho(0,x_0) + \rho(x_0,x)$ by triangle inequality. Note that since A bounded and $x, x_0 \in A$, $\rho(x_0, x) \leq N$ for some $N \geq 0$. Also, A bounded and $x_0 \in A$, so $\rho(0, x_0) \leq M$ for some $M \geq 0$.

$$\begin{aligned} \rho(0,x) &\leq \rho(0,x_0) + \rho(x_0,x) \\ &\leq M + N \\ &= M' && \text{(for some } M' \geq 0) \end{aligned}$$

So for all $x \in A, x \in \prod_{i=1}^{n}[-M', M']$. That is, there is some n-dimension cube, $[-M', M']^n$, bounding any closed and bounded A.

We have that the set $[-M', M']$ is compact (why?), and the finite product of compact spaces is compact.

$\therefore$ Since A is closed by hypothesis, by Theorem 3.3.2, A is compact. □

■ **Example 3.9** The torus $T = S^1 \times S^1$ is compact. It is the product of two compact spaces S^1. By Heine-Borel, S^1 is compact because it is bounded and closed (e.g., show it is equal to its closure, or show we can show $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$ is continuous and $f^{-1}(\{1\}) = S^1$). ■

4. Countability and Separation Axioms

4.1 Countability Axioms

Definition 4.1.1 — Countability Axioms. Let X be a topological space.

(a) Let $x \in X$. We say there exists a *countable base at* x if there exists a countable collection $\{B(x_i)\}_{i \in \mathbb{N}}$ of open sets such that $x \in B(x_i)$ and for any open set U such that $x \in U$, there exists $B_j(x_j)$ such that $x \in B_j(x_j) \subseteq U$.
X has a countable base at each $x \in X$, then X is *first countable*.

(b) X is *second countable* if there is a countable basis that generates $\mathcal{T}_X$.

(c) X is *Lindeloff* if every open cover of X has countable subcover.

A countable base at x is a countable collection of open sets such that for any open neighbourhood U of x in the topology, there is a set within the base that covers x contained in U.

■ **Example 4.1** Every metrizable space is first countable. ■

R Recall the *Archimedean Property*: For every $\varepsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \varepsilon$.

Proof. Say we have metric d. Let $x \in X$ be arbitrary.

Choose $\{B_d(x, \frac{1}{n})\}_{n \in \mathbb{Z}^+}$. It is a countable collection of open sets, and $x \in B_d(x, \frac{1}{n})$.

Let $U \in \mathcal{T}_X = \mathcal{T}_d$ such that $x \in U$. Since U is open, $U = \bigcup_{x \in U} B_d(x, \varepsilon)$ where $\varepsilon > 0$. Then $x \in B_d(x, \varepsilon)$ for some ε. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that $x \in B_d(x, \frac{1}{n}) \subseteq B_d(x, \varepsilon) \subseteq U$. □

Proposition 4.1.1 A second countable space X is first countable and Lindeloff.

4.2 Separation Axioms

Definition 4.2.1 — Separation Axioms. Let X be a topological space.

(a) X is T_0 *or Kolmogorov* if for any distinct $x,y \in X$, there exists open U such that $x \in U$ and $y \notin U$ or $x \notin U$ and $x \in U$.

(b) X is T_1 *or Fréchet* if for any distinct $x,y \in X$, there exists open U,V such that $x \in U$ and $y \notin U$, and $x \notin V$ and $y \in V$.

(c) X is T_2 *or Hausdorff* if for any distinct $x,y \in X$, there exists open U,V such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. (Separated by neighbourhoods.)

(d) X is T_3 *or regular* if for any closed set C and point $x \notin C$, there exists open U,V such that $C \subseteq U$, $x \in V$, and $U \cap V = \varnothing$, and also singleton sets are closed.

(e) X is T_4 *or normal* if for any closed sets C,D such that $C \cap D = \varnothing$, there exists open U,V such that $C \subseteq U$, $D \subseteq V$, and $U \cap V = \varnothing$, and also singleton sets are closed.

These are topological invariants.

Proposition 4.2.1 Every metrizable space is normal.

Proof. Let X be a metrizable space with metric d.

Let C,D be closed, disjoint sets in X.

Let $x \in C$ be arbitrary.

Then there exists $\varepsilon_x > 0$ such that $B(x,\varepsilon_x) \cap D = \varnothing$ because $X - D$ is open and $C \subseteq X - D$.

Similarly, there exists $\varepsilon_y > 0$ such that $B(y,\varepsilon_y) \cap C = \varnothing$.

Choose $U = \bigcup_{x \in C} B(x,\varepsilon_x/2)$ and $V = \bigcup_{y \in D} B(y,\varepsilon_y/2)$. U,V are open since they're the union of open balls, and $C \subseteq U$ and $D \subseteq V$.

Also, U,V are disjoint. For contradiction, suppose there exists $z \in U \cap V$.

Then for some $x \in C$ and $y \in D$, $B(x,\varepsilon_x/2) \cap B(y,\varepsilon_y/2) \neq \varnothing$.

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) && \text{(Triangle inequality)} \\ &< \frac{\varepsilon_x + \varepsilon_y}{2} \\ &\leq \max\{\varepsilon_x, \varepsilon_y\} \\ &\leq \varepsilon_x && (\text{WLOG } \varepsilon_y \leq \varepsilon_x) \end{aligned}$$

So $y \in B(x,\varepsilon_x) \cap D = \varnothing$. Contradiction.

$\therefore U \cap V = \varnothing$, and thus X is normal.

□

Proposition 4.2.2 $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

Part Two

Bibliography

Books

Mun99 Munkres, J. *Topology* 2nd edition (Pearson, Essex, England, 1999).

Index